Lineární algebra

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Těleso $\mathbb{F}$ je množina, $+: F\times F \to F$, $(a,b)\mapsto a+b$, $\cdot: F\times F\to F$, $(a,b)\mapsto ab$

1. existuje $0\in F$: pro $\forall a\in F$, $a+0=0+a=a$
   $(a+b)+c = a+(b+c)$
   $a+b = b+a$
   $\forall a\in F\ \exists b\in F: a+b=0$ $\quad (b=-a)$
2. existuje $1\in F$: $\forall a\in F$ $a\cdot 1 = 1\cdot a = a$
   $(ab)c = a(bc)$
   $ab = ba$
   $\forall a\in F, a\neq 0\ \exists b\in F: ab=1$ $\quad (b=a^{-1})$
3. $a(b+c) = ab+ac$
   $(b+c)a = ba+ca$

nejsou tělesa $\mathbb{N}, \mathbb{Z}$

pro $5\in\mathbb{N},\mathbb{Z}$
$5+(-5)=0$
$-5\notin\mathbb{N}$
$\in\mathbb{Z}$

$5\in\mathbb{Z}$
$5\cdot\frac{1}{5}=1$

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ jsou tělesa

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$ není těleso

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$-1 = 3$

$-2 = 2$

$-3 = 1$

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\mathbb{Z}_p$ je těleso $\Leftrightarrow$ p je prvočíslo

$1^{-1} = 1$

$2^{-1} =$ neex.

$3^{-1} = 3$

$3 \cdot 3 = 1$

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$-1 = 4$

$-2 = 3$

$-3 = 2$

$-4 = 1$

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$1^{-1} = 1$

$2^{-1} = 3$

$3^{-1} = 2$

$4^{-1} = 4$

$2 \cdot 3 = 1, \quad 4 \cdot 4 = 1$

Lineární prostor L nad tělěsem $\mathbb{F}$

$+ : L \times L \to L$

$\cdot : F \times L \to L$

1. existuje $\bar{0} \in L$, $\forall \bar{x} \in L$ $\quad \bar{0} + \bar{x} = \bar{x} + \bar{0} = \bar{x}$ $\qquad 0 \cdot \bar{x} = \bar{0}$

$(\bar{x} + \bar{y}) + \bar{z} = \bar{x} + (\bar{y} + \bar{z})$

$\bar{x} + \bar{y} = \bar{y} + \bar{x}$

$\forall \bar{x} \in L \quad \exists\, \bar{y} \in L : \bar{x} + \bar{y} = \bar{0} \qquad (\bar{y} = -\bar{x})$

2. $\forall \bar{x} \in L \quad 1 \cdot \bar{x} = \bar{x}$

$\forall a, b \in F, \bar{x} \in L \qquad a(b\bar{x}) = (ab)\bar{x}$

z F

3. $\forall a, b \in F \qquad (a+b)\bar{x} = a\bar{x} + b\bar{x}$

$\bar{x}, \bar{y} \in L \qquad a(\bar{x} + \bar{y}) = a\bar{x} + a\bar{y}$

1. ~~$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$~~, $\mathbb{R}$, $\mathbb{C}$ nad $\mathbb{R}$?

$+ : L \times L \to L$

$\cdot : \mathbb{R} \times L \to L$

$\mathbb{R} \times \mathbb{Z} \to \mathbb{R}$

$\pi \cdot 2 = 2\pi \notin \mathbb{Z}$

je prostor

$$\boxed{\begin{gathered} \mathbb{R}^2 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix},\ a, b \in \mathbb{R} \right\} \\ \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix} \\ \alpha \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix} \end{gathered}}$$

$$0 \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \overline{0}$$

$\mathbb{R}^2$ není prostor

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

$$\alpha \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a \\ b \end{pmatrix}$$

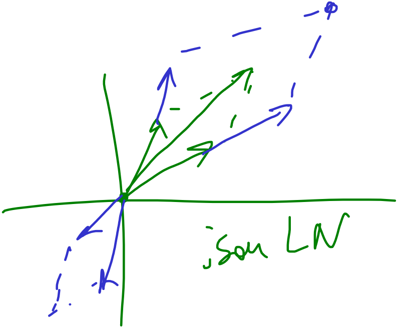
$$0 \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} \neq \overline{0}$$

2. $\mathbb{R}^n$, $\mathbb{C}^n$

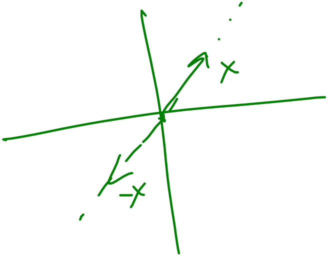
nad $\mathbb{R}$

nad $\mathbb{C}$

3. $\mathbb{R}[x]$, $\mathbb{C}[x]$



jsou LN



$$x+(-x)=0$$

jsou LZ

$$\alpha_1\begin{pmatrix}x_1\\y_1\end{pmatrix}+\alpha_2\begin{pmatrix}x_2\\y_2\end{pmatrix}+\alpha_3\begin{pmatrix}x_3\\y_3\end{pmatrix}=0$$

Lineární kombinace konečného seznamu vektorů S

$S=(x_1, x_2, x_3, \dots x_n)$

seznam koef. $(\alpha_1, \alpha_2, \dots \alpha_n)$

$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$

$\sum_{i=1}^{n} \alpha_i x_i$

triviální $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

nulová $\sum_{i=1}^{n} \alpha_i x_i = \overline{0}$

LN seznam vektorů 1. $S = \emptyset$

2. $S = (x_1, \dots, x_n)$ $\sum_{i=1}^{n} \alpha_i x_i = \overline{0}$, pak $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

LN množina vektorů 1, 2,

3. M je nekonečná každá její konečná podmnožina je LN

$S=(u_1, u_2, u_3, u_1+u_2)$ $\quad u_1, u_2, u_3 \in L$ nad $\mathbb{R}$

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 (u_1+u_2) = \bar{0}$$

$\alpha_1 = 1, \ \alpha_2 = 1, \ \alpha_3 = 0 \quad \alpha_4 = -1$ $\qquad S$ je LZ

$u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ u_2 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \ u_3 = \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}$ v $\mathbb{R}^3$

$\rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = \bar{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ 2\alpha_1 \\ 3\alpha_1 \end{pmatrix} + \begin{pmatrix} 3\alpha_2 \\ -\alpha_2 \\ 2\alpha_2 \end{pmatrix} + \begin{pmatrix} 4\alpha_3 \\ -6\alpha_3 \\ 2\alpha_3 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 + 3\alpha_2 + 4\alpha_3 \\ 2\alpha_1 - \alpha_2 - 6\alpha_3 \\ 3\alpha_1 + 2\alpha_2 + 2\alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \alpha_1 + 3\alpha_2 + 4\alpha_3 &= 0 \\ 2\alpha_1 - \alpha_2 - 6\alpha_3 &= 0 \\ 3\alpha_1 + 2\alpha_2 + 2\alpha_3 &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & -6 \\ 3 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 \\ 0 & -7 & -14 \\ 0 & -7 & -10 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \sim \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix} \begin{array}{l} \\ -\frac{1}{7}R_2 \\ R_3 - R_2 \end{array}$$

$$4\alpha_3 = 0 \Rightarrow \alpha_3 = 0$$

$$\alpha_2 + 2\alpha_3 = 0 \Rightarrow \alpha_2 = 0$$

$$\alpha_1 + 3\alpha_2 + 4\alpha_3 = 0 \Rightarrow \alpha_1 = 0$$

$\mathbb{C}$ nad $\mathbb{R}$ $\quad x = 2+i,\ y = 2-i,\ z = 3+5i$

$\bar{o} = 0 + 0\cdot i$ $\quad (x, y, z)$ jsou ~~LN~~ LZ ?

$$\alpha_1 x + \alpha_2 y + \alpha_3 z = \bar{o}$$

$$\alpha_1(2+i) + \alpha_2(2-i) + \alpha_3(3+5i) = 0$$

$$(2\alpha_1 + 2\alpha_2 + 3\alpha_3) + (\alpha_1 - \alpha_2 + 5\alpha_3)i = 0$$

$$\left.\begin{aligned} 2\alpha_1 + 2\alpha_2 + 3\alpha_3 &= 0 \\ \alpha_1 - \alpha_2 + 5\alpha_3 &= 0 \end{aligned}\right] -2 \qquad \begin{aligned} \alpha_1 - \alpha_2 + 5\alpha_3 &= 0 \\ 0 + 4\alpha_2 - 7\alpha_3 &= 0 \end{aligned}$$

$$\alpha_3 = 4,\quad \alpha_2 = 7 \quad \alpha_1 = 7 - 5\cdot 4 = -13$$

$$-13(2+i) + 7(2-i) + 4(3+5i) = 0$$

$M = \{ \cancel{0}, \underbrace{x+1, x, x^2-x, 3x^2+x+1}_{\text{je LZ/LN}} \} \subseteq \mathbb{R}[x]$

M je $\underline{\underline{LZ}}$/LN ?

$$5 \cdot \bar{0} + 0(x+1) + 0(x) + \dots = 0$$