# Lineární algebra

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$M \subseteq \mathbb{R}^3$ báze $B = \left( \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right)$ LN

$\dim M = 2$

$\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n = \vec{0}$

$\alpha_1 = \alpha_2 = \dots \alpha_n = 0$

Báze a dimenze $M = \left\{ \begin{pmatrix} x + 2y \\ y \\ 2x + y \end{pmatrix} ; \ x, y \in \mathbb{R} \right\}$ nad $\mathbb{R}$.

$$\begin{pmatrix} x+2y \\ y \\ 2x+y \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 2x \end{pmatrix} + \begin{pmatrix} 2y \\ y \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} x + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} y$$

$\alpha$ $\beta$

$x, y \in \mathbb{R}$

$$\alpha \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \alpha + 2\beta &= 0 \\ \beta &= 0 \\ 2\alpha + \beta &= 0 \end{aligned} \Rightarrow \alpha = \beta = 0$$

$(p+q)(-x) = p(-x) + q(-x) = -p(x) - q(x) = -(p+q)(x)$

$(\alpha p)(-x) = \alpha \cdot p(-x) = \alpha \cdot (-p(x)) = -(\alpha p)(x)$

$q(-x) = -q(x)$

Báze a dimenze $M = \{p(x) \in \mathbb{R}^{\leq 3}[x] \; : \; p(-x) = -p(x)\}$.

$(p+q)(x)$

Je to podprostor?

1) $0 \in M$

2) $p(x), q(x) \in M$, pak $p(x) + q(x) \in M$

3) $p(x) \in M$, $\alpha \in \mathbb{R}$, pak $\alpha p(x) \in M$

$\mathbb{R}^{\leq 3}[x]$ má bázi $B = \cancel{(0, 1, 2, 3)}$

$(x^0, x^1, x^2, x^3)$

$a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 = p(x)$

$a_0, a_1, a_2, a_3 \in \mathbb{R}$

$B_M = (x, x^3)$ $\quad \dim M = 2$

$M = \{a_1 x + a_3 x^3, \; a_1, a_3 \in \mathbb{R}\}$

$$P(-x) = a_0 - a_1 x + a_2 x^2 - a_3 x^3 \qquad (-x)^3 = -x^3$$

$$-P(x) = -a_0 - a_1 x - a_2 x^2 - a_3 x^3$$

$$a_0 = -a_0 \Rightarrow 2a_0 = 0, \quad \underline{a_0 = 0}$$

$$\underline{-a_1 = -a_1}$$

$$a_2 = -a_2 \Rightarrow 2a_2 = 0, \quad \underline{a_2 = 0}$$

$$\underline{-a_3 = -a_3}$$

Prostor $L$ nad $F$

Báze $B = (b_1, b_2, \dots, b_n)$

$v \in L$ $\quad v = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$

Souřadnice vektoru $v$ vzhledem k uspořádané bázi $B$

$$\mathrm{coord}_B(v) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$\begin{pmatrix}2\\3\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix}$$

V $\mathbb{R}^2$ nad $\mathbb{R}$.

$$\boxed{\mathrm{coord}_B \begin{pmatrix}2\\3\end{pmatrix}} = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{kde } B = \left(\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right)$$

$1 \cdot b_1 + 0 \cdot b_2$

$\alpha_1 b_1 + \alpha_2 b_2$

$$\begin{pmatrix}2\\3\end{pmatrix} = 1 \cdot \begin{pmatrix}2\\3\end{pmatrix} + 0 \begin{pmatrix}2\\1\end{pmatrix}$$

$$\mathrm{coord}_C \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \quad \text{kde } C = \left(\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}\right)$$

$\alpha_1 c_1 + \alpha_2 \cdot c_2$

$$\begin{pmatrix}2\\3\end{pmatrix} = 0 \begin{pmatrix}2\\1\end{pmatrix} + 1 \begin{pmatrix}2\\3\end{pmatrix}$$

$$\mathrm{coord}_D \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \quad \text{kde } D = \left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right)$$

$$\begin{pmatrix}2\\3\end{pmatrix} = 1 \cdot \begin{pmatrix}1\\1\end{pmatrix} + 1 \cdot \begin{pmatrix}1\\2\end{pmatrix}$$

V $\mathbb{R}^{\leq 3}[x]$ nad $\mathbb{R}$ nalezněte souřadnice

$$p(x) = 6x^3 - 7x + 12$$

vzhledem k bázi $B = (1, x, x^2, x^3)$

$D = (x^3, x^2, x, 1)$

$$\text{coord}_B\, p(x) = \begin{pmatrix} 12 \\ -7 \\ 0 \\ 6 \end{pmatrix} \qquad \text{v } D \begin{pmatrix} 6 \\ 0 \\ -7 \\ 12 \end{pmatrix}$$

$C = (12, 7x, 15x^2, 6x^3)$

$$\text{coord}_C\, p(x) = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$B = (b_1, b_2)$ a $C = (c_1, c_2)$ jsou uspořádané báze prostoru $L$ nad $\mathbb{R}$.

$$\left[\begin{aligned} c_1 &= b_1 - 2b_2 \\ c_2 &= 3b_1 - 5b_2 \end{aligned}\right. \qquad \begin{aligned} b_1 &= \\ b_2 &= \end{aligned}$$

$$u = -4c_1 + 3c_2 =$$
$$= -4(b_1 - 2b_2) + 3(3b_1 - 5b_2)$$
$$= 5b_1 - 7b_2$$

Nechť $u$, $v$ jsou vektory z $L$.

$$\mathrm{coord}_C(u) = \begin{pmatrix} -4 \\ 3 \end{pmatrix}, \qquad \mathrm{coord}_B(u) = ? \begin{pmatrix} 5 \\ -7 \end{pmatrix}$$

$$\mathrm{coord}_B(v) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \qquad \mathrm{coord}_C(v) = ? \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$v = 2b_1 - 3b_2$$
$$v = \alpha c_1 + \beta c_2 = \alpha(b_1 - 2b_2) + \beta(3b_1 - 5b_2) =$$

$$= (\alpha + 3\beta) b_1 + (-2\alpha - 5\beta) b_2$$

$$= 2 b_1 - 3 b_2$$

$$\alpha + 3\beta = 2 \quad / \cdot 2$$

$$-2\alpha - 5\beta = -3$$

$+$

$$\beta = 1$$

$$\alpha = -1$$

$$\beta = 1$$

## Lemma (Exchange Lemma)

Ať $B = (b_1, \ldots, b_n)$ je uspořádaná báze lineárního prostoru $L$ a ať

$$v = \sum_{i=1}^{n} \alpha_i b_i, \qquad \operatorname{coord}_B(v) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

je jakýkoli vektor v $L$. Jestliže $\alpha_i \neq 0$, potom seznam $B[v \leftrightarrow b_i]$, vytvořený z $B$ záměnou $b_i$ za $v$, je opět báze prostoru $L$.

$b_i = v - \frac{1}{\alpha_i} \Sigma$

LN
gen.

$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$ jsou LZ

$\underbrace{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}}_{LN}$

$x+y+w=0$

Nechť $x, y, z, w \in L$ a nechť $\boxed{x + y + z + w = 0.}$

Vysvětlete, proč

a) $\text{span}\{x, y, z\} = \text{span}\{y, z, w\}$ $= \text{span}\{x, y, z, w\}$

$w \overset{?}{\in} \text{span}\{x, y, z\}$, $\boxed{w = -x - y - z}$

$x \in \text{span}\{y, z, w\}$, $x = -y - z - w$

b) pokud $x, y, z$ jsou LN, pak $y, z, w$ jsou LN

$\alpha y + \beta z + \gamma w = \vec{0}$, proč $\alpha = \beta = \gamma = 0$ je jediné řešení?

$\alpha y + \beta z + \gamma(-x - y - z) = 0$

$-\gamma x + (\alpha - \gamma) y + (\beta - \gamma) z = 0$ $\quad z$ LN $\{x, y, z\} \Longrightarrow$

$$-\gamma = 0$$
$$\alpha - \gamma = 0$$
$$\beta - \gamma = 0$$

$$\gamma = 0$$
$$\alpha = 0$$
$$\beta = 0$$

v $\mathbb{R}^3$ podprostory $A$, $B$,

$A \cup B$ není podprostor

$A \cap B$ je podprostor

$$A = \text{span}(a_1, a_2, a_3), \text{ kde } a_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\alpha_1, \ a_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\alpha_2, \ a_3 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}\alpha_3$$

jsou LN?

LZ

báze $(a_1, a_2)$

$\dim A = 2$

$$B = \text{span}(b_1, b_2, b_3), \text{ kde } b_1 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \ b_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \ b_3 = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$$

LZ

báze $(b_1, b_2)$

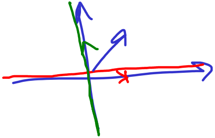
$\dim B = 2$

Najděte bázi a dimenzi $A$, $B$, $A \vee B$, $A \cap B$.

$\text{span}(A \cup B)$

$A \vee B$ 2 nebo 3

$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 b_1 + \alpha_4 b_2 = 0$



$x \in \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$y \in \text{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 3 \end{pmatrix} \sim \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array}$$

$a_1, a_2, a_3$ jsou LZ

$$\beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3 = 0$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 2 & -3 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & -3 \\ 0 & 5 & -5 \\ 0 & 8 & -8 \end{pmatrix} \begin{array}{l} R_3 \\ R_1 + 2R_3 \\ R_2 + 3R_3 \end{array} \sim \begin{pmatrix} -1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & -1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & -3 & 1 \end{pmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \sim$$

$$\boxed{\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 b_1 + \alpha_4 b_2 = 0}$$

$$\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 = 0 \qquad \alpha_1 = -2$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} R_3 - 2R_2$$

$$-\alpha_3 + \alpha_4 = 0$$

$$\alpha_4 = 1$$

$$\alpha_3 = +1, \quad \alpha_2 = 1$$

báze $A \vee B$

$(a_1, a_2, b_1)$

$\dim A \vee B = 3$

$A \cap B$, $\dim A \cap B = 1$

$$\underset{2}{\dim A} + \underset{2}{\dim B} = \underset{3}{\dim A \vee B} + \underset{1}{\dim A \cap B}$$

$$b_1 + b_2 = 2a_1 + a_2$$

báze $A \cap B$ je $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$ $\quad \alpha_1 = -2,\ \alpha_2 = -1,\ \alpha_3 = 1,\ \alpha_4 = 1$

$$b_1 + b_2 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$$

$$\cancel{\begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix} \in A \cap B} \subseteq \mathbb{R}^3$$

$$2a_1 + a_2 = 2\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$$

$r \in A \cap B$

$$r = \underbrace{\beta_1 a_1 + \beta_2 a_2}_{} = \underbrace{\beta_3 b_1 + \beta_4 b_2}_{}$$

# Matice

lin. zobr.

Spočtěte $A+B$, $-A$, $A^T$, $AB$, $BA$ pro

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{pmatrix}.$$