Lineární algebra

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Nechť $G : Mat_{2x2} \to \mathbb{R}$ je zobrazení určené předpisem

$$G\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + b + c + d$$

Dokažte, že se jedná o lineární zobrazení. Určete jeho jádro a obraz.

$\dim \operatorname{Ker} G = 3$, $\operatorname{Ker} G = \operatorname{span}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\right)$

$\operatorname{Ker} G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a+b+c+d = 0 \right\}$

$d = -a-b-c$

$a = 1,\ b = c = 0,\ d = -1$

$b = 1,\ a = c = 0,\ d = -1$

$c = 1,\ a = b = 0,\ d = -1$

$f(\alpha u+\beta v+\gamma w) = \alpha f(u)+\beta f(v)+\gamma f(w)$

Mějme $f\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$, $f\begin{pmatrix}-3\\4\\-2\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$, $f\begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}-3\\5\end{pmatrix}$.

u, v, w

1) Dokažte, že existuje právě jedno lineární zobrazení $f : \mathbb{R}^3 \to \mathbb{R}^2$ s těmito předpisy.

2) Zjistěte jaké je jeho jádro a obraz.

$\operatorname{Ker} f = \{x \mid f(x) = 0\}$

$x = \alpha u+\beta v+\gamma w$

$f(x) = \alpha f(u)+\beta f(v)+\gamma f(w) = \begin{pmatrix}0\\0\end{pmatrix}$

3) Určete hodnotu $f\begin{pmatrix}-1\\2\\1\end{pmatrix}$ a

4) najděte všechna $v_1 \in \mathbb{R}^3$ pro která $f(v_1) = \begin{pmatrix}-1\\4\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}-3\\5\end{pmatrix} - \begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}2\\-1\end{pmatrix}$

$f(v)$ $f(w)$

1) u, v, w jsou $\boxed{LN}$/LZ?, $\alpha u+\beta v+\gamma w = 0$

$v_1 = v+w = \begin{pmatrix}-3\\6\\-1\end{pmatrix} + \operatorname{ker} f$

$a \in \mathbb{R}$ $\quad a\begin{pmatrix}-3\\6\\-1\end{pmatrix} = av + aw$, $f(av + aw) =$

$= a f(v+w)$

$= a \begin{pmatrix}-1\\4\end{pmatrix}$

$f(a \cdot v_1) = a\begin{pmatrix}-1\\4\end{pmatrix}$

$f(v_1) = \begin{pmatrix}-1\\4\end{pmatrix}$, $w_1 \in \ker f$

$f(v_1 + w_1) = f(v_1) + f(w_1) = \begin{pmatrix}-1\\4\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}-1\\4\end{pmatrix}$

$$\operatorname{Im} f = \{ f(x) \mid x \in \mathbb{R}^3 \} =$$

$$= \operatorname{span}(f(u), f(v), f(w)) = \operatorname{span}\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right)$$

$$= \mathbb{R}^2$$

$$\alpha \begin{pmatrix}1\\3\end{pmatrix} + \beta \begin{pmatrix}2\\-1\end{pmatrix} + \gamma \begin{pmatrix}-3\\5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & -3\\ 3 & -1 & 5\end{pmatrix} = \begin{pmatrix}1 & 2 & -3\\ 0 & -7 & 14\end{pmatrix} \; R_2 - 3R_1 \; = \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -2\end{pmatrix} \; \begin{matrix} R_1 - 2R_2' \\ \frac{R_2}{-7} R_2' \end{matrix}$$

$$\alpha + \gamma = 0 \qquad \alpha = -\gamma$$
$$\beta - 2\gamma = 0 \qquad \beta = 2\gamma, \quad \gamma \in \mathbb{R} \qquad \gamma = 1$$

$$\begin{pmatrix}0\\0\end{pmatrix} = -\gamma \begin{pmatrix}1\\3\end{pmatrix} + 2\gamma \begin{pmatrix}2\\-1\end{pmatrix} + \gamma \begin{pmatrix}-3\\5\end{pmatrix}$$

$$x = -\gamma \begin{pmatrix}1\\0\\2\end{pmatrix} + 2\gamma \begin{pmatrix}-3\\4\\-2\end{pmatrix} + \gamma \begin{pmatrix}0\\2\\1\end{pmatrix} = \gamma \underline{\underline{\begin{pmatrix}-7\\10\\-5\end{pmatrix}}}$$

$$\operatorname{Ker} f = \operatorname{span}\left(\begin{pmatrix}-7\\10\\-5\end{pmatrix}\right)$$

$$\alpha \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 4 & 2 \\ 2 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 0 \\ 0 & 4 & 2 \\ 0 & 4 & 1 \end{pmatrix}_{R_3 - 2R_1} \sim \begin{pmatrix} 1 & -3 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix}_{R_2 - R_3}$$

$$\Rightarrow \alpha = \beta = \gamma = 0$$

$$\alpha \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & -1 \\ 0 & 4 & 2 & 2 \\ 2 & -2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 0 & -1 \\ 0 & 4 & 2 & 2 \\ 0 & 4 & 1 & 3 \end{array}\right)_{R_3 - 2R_1} \sim \left(\begin{array}{ccc|c} 1 & -3 & 0 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)^{\frac{1}{2}R_2}_{R_2 - R_3}$$

$\gamma = -1$

$2\beta + \gamma = 1 \;,\quad \beta = 1$

$\alpha - 3\beta = -1 \;,\quad \alpha = 2$

$$\underset{\alpha}{2u} + \underset{\beta}{v} - \underset{\gamma}{w} = \boxed{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}}$$

$$f\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = f(2u + v - w) = 2\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$