Lineární algebra

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Maticové rovnice

$AA^{-1} = A^{-1}A = E$

$AX = B$

$(A|B) \sim (E|X)$

$AA^{-1} = E$

$(A|E) \sim (E|A^{-1})$

Nad $\mathbb{R}$ vyřešte $\overbrace{\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}}^{A} X = \overbrace{\begin{pmatrix} 4 & -6 \\ 2 & 1 \end{pmatrix}}^{B}$

$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$2a + 5c = 4$
$a + 3c = 2$

$2b + 5d = -6$
$b + 3d = 1$

$$\left(\begin{array}{cc|c|c} 2 & 5 & 4 & -6 \\ 1 & 3 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c|c} 1 & 3 & 2 & 1 \\ 0 & -1 & 0 & -8 \end{array}\right) \begin{array}{l} R_2 \\ R_1 - 2R_2 \end{array} \sim \left(\begin{array}{cc|c|c} 1 & 0 & 2 & -23 \\ 0 & 1 & 0 & 8 \end{array}\right) \begin{array}{l} R_1 + 3R_2 \\ -R_2 \end{array}$$

a, c $b\ d$ X

$a = 2$ $b = -23$
$c = 0$ $d = 8$

$$X = \begin{pmatrix} 2 & -23 \\ 0 & 8 \end{pmatrix}$$

$\rightarrow AX = B$, $X = \cancel{\frac{B}{A}}$ BA^{-1}

$A^{-1}B$

$A^{-1}(AX) = A^{-1}B$

$(A^{-1}A)X = A^{-1}B$

$EX = A^{-1}B$

$\rightarrow X = A^{-1}B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 & -6 \\ 2 & 1 \end{pmatrix} = \underline{\begin{pmatrix} 2 & -23 \\ 0 & 8 \end{pmatrix}}$

$\left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right)$

$A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{array}\right) \sim \underbrace{\left(\begin{array}{cc|cc} 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)}_{E}$$

Nad $\mathbb{R}$ vyřešte $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} X = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$ $\quad X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$2a + c = 2 \qquad 2b + d = 1$$
$$2a + c = 2 \qquad 2b + d = 1$$

$$c = 2 - 2a \qquad d = 1 - 2b$$

$a = b = 0$ $\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

$$X = \begin{pmatrix} a & b \\ 2-2a & 1-2b \end{pmatrix}, \quad a, b \in \mathbb{R}$$

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Nad $\mathbb{R}$ vyřešte $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$2a + c = 1 \qquad 2b + d = 0$

$2a + c = 0 \qquad 2b + d = 1$

nemá řešení

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right)_{R_2-2R_1} \sim \left(\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right)^{R_1-R_2}$$

Nad $\mathbb{R}$ vyřešte $\quad A - XA = E$, kde $\quad A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

$$-XA = E - A$$

$$XA = A - E$$

$$(XA)A^{-1} = (A-E)A^{-1}$$

$$X = AA^{-1} - EA^{-1} = E - A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & 0 \end{pmatrix}$$

$\dim \mathbb{R}^{\leq 4}[x] = 5$

báze je $(x^4, x^3, x^2, x, 1)$

Spočtěte jádro, obraz, defekt a hodnost lineárního zobrazení

$der: \mathbb{R}^{\leq 4}[x] \to \mathbb{R}^{\leq 4}[x]$

$$\underbrace{(ax^4 + bx^3 + cx^2 + dx + e)}_{p(x)} \mapsto \underbrace{(4ax^3 + 3bx^2 + 2cx + d)}_{p'(x)}$$

$\mathrm{Ker}(der) = \{p(x) \in \mathbb{R}^{\leq 4}[x] \mid p'(x) = 0\} = \{e \mid e \in \mathbb{R}\}$

$\boxed{4ax^3 + 3bx^2 + 2cx + d = 0}$

$a = 0$ $\quad 4a = 0$

$b = 0$ $\quad 3b = 0$

$c = 0$ $\quad 2c = 0$

$d = 0$ $\quad d = 0$

$\mathrm{def}(der) = 1$

$\mathrm{Im}(der) = \mathrm{span}(x^3, x^2, x, 1)$

$\mathrm{rank}(der) = 4$

Matice lineárního zobrazení

$f: L_1 \to L_2$, B báze L_1, C báze L_2

$B = (b_1, b_2, \dots b_n)$

$M_{BC} = (\mathrm{coord}_C f(b_1), \mathrm{coord}_C f(b_2), \dots)$

Najděte matici lin.zobrazení *der* vzhledem

- k bázi $B = (x^4, x^3, x^2, x, 1)$,
- k bázi $C = (1, x, x^2, x^3, x^4)$,
- k bázím $D = ((x-2)^4, (x-2)^3, (x-2)^2, (x-2), 1)$ a $B = (x^4, x^3, x^2, x, 1)$.

M_{BC}, M_{CB}

$$M_{BB} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad M_{CC} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$b_1 = x^4$, $\mathrm{der}\, x^4 = 4x^3$, $\mathrm{coord}_B(4x^3) = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $b_2 = x^3$

$$M_{DB} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ -24 & 3 & 0 & 0 & 0 \\ 48 & -12 & 2 & 0 & 0 \\ -32 & 13 & -4 & 1 & 0 \end{pmatrix} \qquad M_{DD} = M_{BB}$$

$$\mathrm{dn}\left((x-2)^4\right) = 4(x-2)^3 = 4\left(x^3 + 3x^2(-2) + 3x(-2)^2 + (-2)^3\right)$$

$$= 4\left(x^3 - 6x^2 + 12x - 8\right)$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$$\mathrm{dn}\left((x-2)^3\right) = 3(x-2)^2$$

$$= 3\left(x^2 - 4x + 4\right)$$

$$\mathrm{dn}\left((x-2)^2\right) = 2(x-2)$$

$$\begin{pmatrix}1\\3\end{pmatrix} = \cancel{\alpha}^{2}\begin{pmatrix}2\\-1\end{pmatrix} + \cancel{\beta}^{1}\begin{pmatrix}-3\\5\end{pmatrix}$$

$$f(b_2) = \begin{pmatrix}2\\-1\end{pmatrix}, \quad coord_C\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$f(b_3) = \begin{pmatrix}-3\\5\end{pmatrix}, \quad coord_C\begin{pmatrix}-3\\5\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

Najděte matici $f : \mathbb{R}^3 \to \mathbb{R}^2$

$$\underbrace{f\begin{pmatrix}1\\0\\2\end{pmatrix}}_{b_1} = \begin{pmatrix}1\\3\end{pmatrix}, \quad \underbrace{f\begin{pmatrix}-3\\4\\-2\end{pmatrix}}_{b_2} = \begin{pmatrix}2\\-1\end{pmatrix}, \quad \underbrace{f\begin{pmatrix}0\\2\\1\end{pmatrix}}_{b_3} = \begin{pmatrix}-3\\5\end{pmatrix}$$

- vzhledem k $B = (\begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}-3\\4\\-2\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix})$ a $K_2 = (\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix})$,
- vzhledem k B a $C = (\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-3\\5\end{pmatrix})$.

$$M_{BK_2} = \begin{pmatrix}1 & 2 & -3\\3 & -1 & 5\end{pmatrix} \qquad M_{BC} = \begin{pmatrix}2 & 1 & 0\\1 & 0 & 1\end{pmatrix}$$

$\dim \mathbb{R}^{\leq 1}[x] = 2$, rank $g = 2$, def $g = 0$

báze $\{x, 1\}$

Spočtěte jádro, obraz, defekt a hodnost lineárního zobrazení

$$g : \mathbb{R}^{\leq 1}[x] \to \mathbb{R}^{\leq 1}[x] : g(x+1) = x+2,\ g(2x+1) = 3x.$$

$x+1$, $2x+1$ jsou LN

$\alpha(x+1) + \beta(2x+1) = 0$

$(\alpha+2\beta)x + \alpha+\beta = 0$

$\alpha + 2\beta = 0$ $\quad \beta = 0$

$\alpha + \beta = 0$ $\quad \alpha = 0$

$\operatorname{Im} g = \operatorname{span}(x+2, 3x) = \mathbb{R}^{\leq 1}[x]$

$x+2$, $3x$ jsou LN

$\operatorname{Ker} g = \{0\}$ $\quad$ g je izomorfismus $\quad$ lineární bijekce

$g^{-1} : \mathbb{R}^{\leq 1}[x] \to \mathbb{R}^{\leq 1}[x] :$ $\quad g^{-1}(x+2) = x+1$

$g : L_1 \to L_2$ $\quad g^{-1}(3x) = 2x+1$

$g^{-1} : L_2 \to L_1$

$$M_{BK} = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$$

$g(x+1) = x+2$

$g(2x+1) = 3x$

$\mathrm{coord}_K(x+2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\mathrm{coord}_K(3x) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

Najděte matici lin.zobrazení g vzhledem

- k bázím $B = (\underbrace{x+1}_{b_1}, \underbrace{2x+1}_{b_2})$ a $K = (x, 1)$,
- k bázi K,
- k bázi B.

$x = \alpha_1 b_1 + \alpha_2 b_2$

$1 = \beta_1 b_1 + \beta_2 b_2$

$\mathrm{coord}_B\, x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\mathrm{coord}_B\, 1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$M_{KK} = \begin{pmatrix} 2 & -1 \\ -2 & 4 \end{pmatrix}$$

$$g(x) = g\big(2x+1-(x+1)\big) = g(2x+1) - g(x+1) = 3x - (x+2) = 2x-2$$

$$g(1) = g\big(\underbrace{2}_{\beta_1}(x+1) - \underbrace{(2x+1)}_{\beta_2}\big) = 2g(x+1) - g(2x+1) = 2(x+2) - 3x = -x+4$$

$$1 = \beta_1 (x+1) + \beta_2 (2x+1)$$

$$0 \cdot x + 1 = (\beta_1 + 2\beta_2) x + \beta_1 + \beta_2$$

$$\begin{aligned} \beta_1 + 2\beta_2 &= 0 \\ \beta_1 + \beta_2 &= 1 \end{aligned} \qquad \begin{aligned} \beta_2 &= -1 \\ \beta_1 &= 2 \end{aligned}$$

L, baze $B = (b_1, b_2, \dots, b_n)$, $C = (c_1, c_2, \dots, c_n)$

$$T_{B \to C} \cdot coord_B(\vec{u}) = coord_C(\vec{u})$$

$$T_{B \to C} = \left(coord_C(b_1), coord_C(b_2), \dots \right)$$

$$B = (x+1,\ 2x+1) \qquad T_{B\to K} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$K = (x, 1) \qquad T_{K\to B} = \left(T_{B\to K}\right)^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\mathrm{coord}_K(x+1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathrm{coord}_K(2x+1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right)_{R_1-R_2} \sim \left(\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right)^{R_1-2R_2}$$

$$K \xrightarrow{T_{K\to B}} B \xrightarrow{M_{BK}} K$$

$$M_{KK} = M_{BK}\, T_{K\to B} = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 & -1 \\ -2 & 4 \end{pmatrix}}}$$

$$\operatorname{coord}_K(ax+b) = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \operatorname{coord}_K g(ax+b) = M_{KK} \cdot \operatorname{coord}_K(ax+b)$$

$$= \begin{pmatrix} 2 & -1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a-b \\ -2a+4b \end{pmatrix}$$

Najděte předpis pro zobrazení g, t.j. $g(ax+b) = a\,g(x) + b\,g(1) =$

$$= a\,(2x-2) + b\,(-x+4) = \underline{(2a-b)\,x + (-2a+4b)}$$

Najděte matici lin.zobrazení g^{-1} vzhledem k bázi K.

$$(M_{KK})^{-1} = \begin{pmatrix} 2 & -1 \\ -2 & 4 \end{pmatrix}^{-1} = \;\ldots\; = \begin{pmatrix} \frac{4}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ -2 & 4 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 0 & 3 & 1 & 1 \end{array}\right) \underset{R_1+R_2}{} \sim \left(\begin{array}{cc|cc} 6 & 0 & 4 & 1 \\ 0 & 3 & 1 & 1 \end{array}\right) \underset{3R_1+R_2}{}$$

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -5 & -3 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$B = (b_1, b_2)$ a $C = (c_1, c_2)$ jsou uspořádané báze prostoru L nad $\mathbb{R}$.

$$c_1 = b_1 - 2b_2$$
$$c_2 = 3b_1 - 5b_2$$

$$T_{C \to B} = \begin{pmatrix} 1 & 3 \\ -2 & -5 \end{pmatrix}$$

$\operatorname{coord}_B c_1$ $\operatorname{coord}_B c_2$

Nechť u, v jsou vektory z L.

$\operatorname{coord}_C(u) = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$, $\operatorname{coord}_B(u) = ?$ $\begin{pmatrix} 1 & 3 \\ -2 & -5 \end{pmatrix}\begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}$

$\operatorname{coord}_B(v) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\operatorname{coord}_C(v) = ?$ $T_{B \to C}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -5 & -3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$u = -4c_1 + 3c_2 = -4(b_1 - 2b_2) + 3(3b_1 - 5b_2) = 5b_1 - 7b_2$$