Matematická analýza 2

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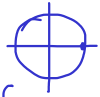
$$\int_{C_2} 1\,ds = 4a\int_0^{\frac{\pi}{2}} \sqrt{4\sin^2 t + \underbrace{\tfrac14 \cos^2 t}_{\geq 0}}\,dt \geq 4a\int_0^{\frac{\pi}{2}} \sqrt{4\sin^2 t}\,dt$$

$$= 8a\int_0^{\frac{\pi}{2}} \sin t\,dt = 8a\,[-\cos t]_0^{\frac{\pi}{2}} = 8a$$

Rozhodněte která křivka má větší délku

a) kružnice o poloměru a,

b) elipsa s poloosami $a/2$, $2a$.

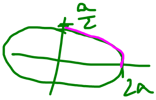


C_1

$$\varphi(t) = (a\cos t,\ a\sin t),\ t\in\langle 0, 2\pi\rangle$$

$$\varphi'(t) = (-a\sin t,\ a\cos t)$$

$$\|\varphi'(t)\| = a \qquad \int_{C_1} 1\,ds = \int_0^{2\pi} a\,dt = 2\pi a < 8a$$



C_2

$$\psi(t) = \left(2a\cos t,\ \tfrac{a}{2}\sin t\right),\ t\in\langle 0, 2\pi\rangle$$

$$\psi'(t) = \left(-2a\sin t,\ \tfrac{a}{2}\cos t\right)$$

$$\|\psi'(t)\| = a\sqrt{4\sin^2 t + \tfrac14\cos^2 t}$$

$$\int_{C_2} 1\,ds = a\int_0^{2\pi} \sqrt{4\sin^2 t + \tfrac14 \cos^2 t}\,dt$$

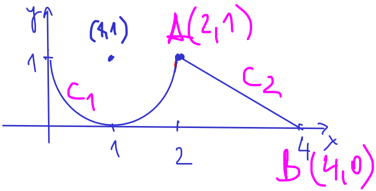
DÚ

$$\int_{C_1\cup C_2} \tau\, ds = \int_{C_1} \tau\, ds + \int_{C_2} \tau\, s \qquad 3\pi-4 + \frac{29}{3}\sqrt{5}$$

$$\int_C f\, ds = \int_a^b f(\varphi)\cdot\|\varphi'\| \qquad \varphi : \langle a,b\rangle \to C$$

Vypočtěte hmotnost křivky C nakreslené na obrázku, jestliže jeji délková hustota v bodě $(x,y)\in C$ je dána funkci $\tau(x,y) = x^2+y^2$.

C_1 parametrizace: $x = \cos t + 1$, $y = \sin t + 1$, $t \in \langle \pi, 2\pi\rangle$



$$\varphi(t) = (1+\cos t,\ 1+\sin t)$$

$$\varphi'(t) = (-\sin t,\ \cos t)$$

$$\|\varphi'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\tau(\varphi(t)) = (1+\cos t)^2 + (1+\sin t)^2 = 1 + 2\cos t + \cos^2 t + 1 + 2\sin t + \sin^2 t$$

$$= 3 + 2\cos t + 2\sin t$$

$$\int_{C_1} \tau\, ds = \int_\pi^{2\pi} (3 + 2\cos t + 2\sin t)\cdot 1\, dt = \left[3t + 2\sin t - 2\cos t\right]_\pi^{2\pi} =$$

$\sin 2\pi = 0$, $\sin \pi = 0$

$$= 3\pi - 2(\underbrace{\cos 2\pi - \cos\pi}_{1-(-1)}) = 3\pi - 4$$

C_2 parametrizace: $\varphi(t) = A + t(B-A), \ t \in \langle 0,1 \rangle$

$$\varphi(t) = (2,1) + t(2,-1)$$

$$\varphi(t) = (2+2t, 1-t)$$

$$\varphi'(t) = (2,-1)$$

$$\|\varphi'(t)\| = \sqrt{4+1} = \sqrt{5}$$

$$\int_{C_2} \tau \, ds = \int_0^1 \Big(\underbrace{(2+2t)^2 + (1-t)^2}_{5+6t+5t^2} \Big) \cdot \sqrt{5} \, dt = \sqrt{5} \left[5t + 3t^2 + \frac{5}{3}t^3 \right]_0^1 =$$

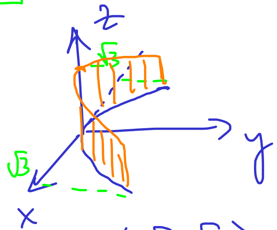
$$= \sqrt{5}\left(5+3+\frac{5}{3}\right) = \frac{29}{3}\sqrt{5}$$

$$\int_C \tau \, ds = 3\pi - 4 + \frac{29}{3}\sqrt{5}$$

Základna plotu je křivka $\underbrace{y = \frac{1}{2}x^2,\ x \in \langle -\sqrt{3}, \sqrt{3} \rangle}_{C}$, výška plotu nad bodem (x, y) je

$$v(x, y) = \frac{1}{1+x^2}.$$

Určete plochu plotu.



$$\int_C v\,ds = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{1+t^2} \cdot \sqrt{1+t^2}\,dt =$$

$$\boxed{\begin{aligned} &\text{parametrizace } C:\ \varphi(t) = \left(t, \tfrac{1}{2}t^2\right),\ t \in \langle -\sqrt{3}, \sqrt{3} \rangle \\ &\varphi'(t) = (1, t),\quad \|\varphi'(t)\| = \sqrt{1+t^2} \\ &v(\varphi(t)) = \frac{1}{1+t^2} \end{aligned}}$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{\sqrt{1+t^2}}\,dt =$$

$$\left|\begin{aligned} &t = \sinh u,\ \operatorname{tg} u,\ \operatorname{cotg} u \\ &\cosh^2 u - \sinh^2 u = 1 \\ &dt = \cosh u\,du \\ &\cosh u = \frac{e^u + e^{-u}}{2} \geq 0 \end{aligned}\right| = \int \frac{\cancel{\cosh u}\,du}{\cancel{\cosh u}} = u = \operatorname{argsinh} t$$

$$= \left[\ln\left(t + \sqrt{t^2+1}\right)\right]_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \ln(\sqrt{3}+2) - \ln(-\sqrt{3}+2) = \ln\left(\frac{\sqrt{3}+2}{-\sqrt{3}+2} \cdot \frac{\sqrt{3}+2}{\sqrt{3}+2}\right) =$$

$$(-a+b)(a+b) = b^2 - a^2$$

$$= 4 - 3 = 1$$

$$= \ln \frac{(\sqrt{3}+2)^2}{1} = 2\ln(\sqrt{3}+2)$$

BONUS

$$\int \frac{1}{\sqrt{1+t^2}}\,dt = \left| \begin{array}{l} t = \sinh u \\ dt = \cosh u\,du \end{array} \right| = \int \frac{\cosh u\,du}{\cosh u} = \int du =$$

$\cosh^2 u - \sinh^2 u = 1$, $\cosh u = \frac{e^x + e^{-x}}{2} \geq 1$

$$= u = \operatorname{arg\,sinh} t = \ln\left(t + \sqrt{t^2+1}\right)$$

$t = \frac{e^u - e^{-u}}{2} \rightsquigarrow u = ?$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{\sqrt{1+t^2}}\,dt = 2\left[\ln\left(t+\sqrt{t^2+1}\right)\right]_0^{\sqrt{3}} = 2\ln\left(\sqrt{3}+2\right)$$

po $e^u = z$, $t = \frac{z - \frac{1}{z}}{2} = \frac{z^2-1}{2z}$, $z^2 - 1 = 2zt$, $z^2 - (2t)z - 1 = 0$

$D = 4t^2 + 4$

$z_{1,2} = \frac{2t \pm 2\sqrt{t^2+1}}{2}$

$t - \sqrt{t^2+1} < 0$

$e^u = t + \sqrt{t^2+1}$

$u = \ln\left(t + \sqrt{t^2+1}\right)$

BONUS

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{\sqrt{1+t^2}}\,dt = \left| \begin{array}{l} t = tg\,u,\ u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ dt = \frac{1}{\cos^2 u}\,du \end{array} \right| = 2\int_0^{\operatorname{arctg}\sqrt{3}} \frac{\cos u}{\cos^2 u}\,du$$

$$\sqrt{1+tg^2 u} = \sqrt{1+\left(\frac{\sin u}{\cos u}\right)^2} = \sqrt{\frac{\cos^2 u+\sin^2 u}{\cos^2 u}} = \sqrt{\frac{1}{\cos^2 u}} = \frac{1}{\cos u},\ u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

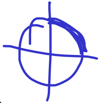
$$= 2\int_0^{\operatorname{arctg}\sqrt{3}} \frac{\cos u}{1-\sin^2 u}\,du = \left| \begin{array}{l} v = \sin u \\ dv = \cos u\,du \end{array} \right| = 2\int_0^{\frac{\sqrt{3}}{2}} \frac{dv}{1-v^2} =$$

$$\cos u = \frac{1}{\sqrt{1+tg^2 u}} = \frac{1}{\sqrt{1+(\sqrt{3})^2}} = \frac{1}{2} \Rightarrow \sin u = \frac{\sqrt{3}}{2}$$

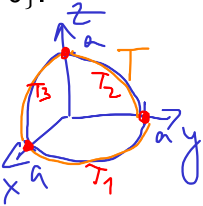
$$= 2\cdot\frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}} \left(\frac{1}{v+1} - \frac{1}{v-1}\right)dv = \Big[\ln|v+1| - \ln|v-1|\Big]_0^{\frac{\sqrt{3}}{2}} = \left[\ln\frac{1+v}{1-v}\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \ln\frac{2+\sqrt{3}}{2-\sqrt{3}}\cdot\frac{2+\sqrt{3}}{2+\sqrt{3}} = 2\ln(2+\sqrt{3})$$

$\rho = 1$



Nalezněte souřadnice těžiště homogenního obvodu sférického trojúhelníku $T = \{(x, y, z) \mid x^2 + y^2 + z^2 = a^2, x, y, z \geq 0\}$.



těžiště $\alpha(1,1,1) = (x_T, y_T, z_T)$

$$x_T = \frac{\int_T x\rho\, ds}{\int_T \rho\, ds} = \frac{\int_T x\, ds}{\boxed{\int_T 1\, ds}} = \frac{2a^2}{\frac{3}{2}\pi a} = \frac{4a}{3\pi}$$

$$\frac{3}{4}\cdot 2\pi a = \frac{3}{2}\pi a$$

$$\int_T x\, ds = \overbrace{\int_{T_1} x\, ds}^{a^2} + \overbrace{\int_{T_2} x\, ds}^{0} + \overbrace{\int_{T_3} x\, ds}^{a^2} = 2a^2$$

T_1: $\boxed{x = a\cos t}$, $y = a\sin t$, $z = 0$

T_2: $x = 0$, $y = a\cos t$, $z = a\sin t$

T_3: $\boxed{x = a\cos t}$, $y = 0$, $z = a\sin t$

pro $t \in \langle 0, \frac{\pi}{2} \rangle$

T_1: $\varphi(t) = (a\cos t, a\sin t, 0)$

$\varphi'(t) = (-a\sin t, a\cos t, 0)$, $\|\varphi'(t)\| = a$

$$\int_{T_1} x\,ds = \int_0^{\frac{\pi}{2}} a\cos t \cdot a\,dt = a^2 [\sin t]_0^{\frac{\pi}{2}} = a^2$$

$$\frac{4a}{3\pi}(1,1,1)$$

$$a^2\int_0^{\frac{\pi}{2}} \sin t\,dt = a^2 [-\cos t]_0^{\frac{\pi}{2}} = a^2$$

$I=\int_C v^2 \rho \, ds$, kde v je vzdálenost od osy z

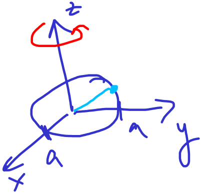
$\rho = 1$

Nalezněte moment setrvačnosti homogenní kružnice o poloměru *a* vzhledem k přímce procházející jejím středem

a)kolmé na rovinu kružnice

b)ležící v rovině kružnice

a)



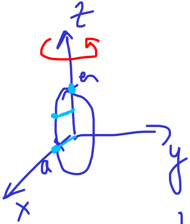
$x^2+y^2=a^2$

kolem osy z

$v = a$

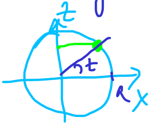
$$\int_C v^2 ds = \int_C a^2 ds = a^2 \underbrace{\int_C ds}_{2\pi a} = 2\pi a^3$$

b)



$x^2+z^2=a^2$

kolem osy z



$v = a\cos t$

parametrizace

$$\left.\begin{aligned} x &= a\cos t \\ y &= 0 \\ z &= a\sin t \end{aligned}\right] \varphi(t)$$

$t \in \langle 0, 2\pi \rangle$

$\|\varphi'(t)\| = a$

$$I_\beta = \int_{C_2} v^2\, ds = \int_0^{2\pi} a^2 \cos^2 t \cdot a\, dt = a^3 \int_0^{2\pi} \cos^2 t\, dt =$$

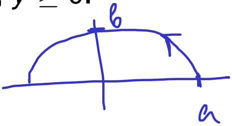
$$= \frac{a^3}{2} \int_0^{2\pi} (1 + \underbrace{\cos 2t}_{\text{periodo } \pi})\, dt = \frac{a^3}{2} \cdot 2\pi = \pi a^3$$

$$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$$

$\vec F: \mathbb{R}^2 \to \mathbb{R}^2$, $\int_{(C)} \vec F\, d\vec s = \int_\alpha^\beta \underbrace{\vec F(\varphi)\cdot \varphi'}_{\text{skalární součin.}}\, dt$ $\quad \varphi: \langle \alpha, \beta \rangle \to C$

Nechť $\vec F(x,y) = (x^2, xy)$. Vypočtěte $\int_{(C)} \vec F d\vec s$, kde C je kladně orientovaná horní část elipsy $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$, $y \geq 0$.

Parametrizace C



$$\varphi(t) = (\overset{x}{\overbrace{a\cos t}}, \overset{y}{\overbrace{b\sin t}}),\ t \in \langle 0, \pi \rangle$$

$$\varphi(0) = (a, 0),\quad \varphi\left(\frac{\pi}{2}\right) = (0, b)$$

$$\varphi'(t) = (-a\sin t, b\cos t)$$

$$\vec F(\varphi(t)) = (a^2\cos^2 t, ab\cos t\sin t)$$

$$\int_{(C)} \vec F d\vec s = \int_0^\pi (a^2\cos^2 t, ab\cos t\sin t)\cdot(-a\sin t, b\cos t)\, dt$$

$$= \int_0^\pi (-a^3\cos^2 t\sin t + ab^2\cos^2 t\sin t)\, dt = (ab^2 - a^3)\int_0^\pi \cos^2 t\sin t\, dt$$

$$= \left| \begin{array}{ll} u = \cos t & \\ du = -\underbrace{\sin t\, dt} & \\ t = 0, & u = 1 \\ t = \pi, & u = -1 \end{array} \right| = (ab^2 - a^3) \int_1^{-1} u^2 (-du) =$$

$$= (ab^2 - a^3) \int_{-1}^{1} u^2 du =$$

$$= (ab^2 - a^3) \left[\frac{u^3}{3} \right]_{-1}^{1} = a(b^2 - a^2) \left(\frac{1}{3} - \left(\frac{-1}{3} \right) \right)$$

$$= \frac{2}{3} a (b^2 - a^2)$$

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$

Určete $\int_{(C)} \vec{F} d\vec{s}$, kde $\vec{F} = (x, y, x + y - 1)$ a (C) je orientovaná úsečka s počátečním bodem A $(1, 1, 1)$ a koncovým bodem B $(2, 3, 4)$.

Parametrizace C: $\varphi(t) = A + t(B - A)$, $t \in \langle 0, 1 \rangle$

$\varphi(t) = (1,1,1) + t(1,2,3) = (\underbrace{1+t}_{x}, \underbrace{1+2t}_{y}, \underbrace{1+3t}_{z})$

$\varphi'(t) = (1,2,3)$

$\vec{F}(\varphi(t)) = (1+t, 1+2t, \underbrace{\overbrace{1+t}^{x} + \overbrace{1+2t}^{y} - 1}_{1+3t})$

$$\int_{(C)} \vec{F} d\vec{s} = \int_0^1 (1+t, 1+2t, 1+3t) \cdot (1, 2, 3)\, dt =$$

$$= \int_0^1 (1+t+2+4t+3+9t)\, dt = \int_0^1 (6+14t)\, dt =$$

$$= \left[6t + 14 \cdot \frac{t^2}{2}\right]_0^1 = 6 + 7 = 13$$

Nalezněte práci vykonanou silovým polem, které směřuje k počátku souřadnicového systému a jehož velikost je (a) přímo, (b) nepřímo úměrná vzdálenosti od počátku. Bod se pohybuje po elipse o rovnici $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ a to od bodu $(a, 0)$ k bodu $(0, b)$.

DÚ