Matematická analýza 2

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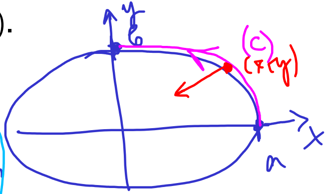
Nalezněte práci vykonanou silovým polem, které směřuje k počátku souřadnicového systému a jehož velikost je (a) přímo, (b) nepřímo úměrná vzdálenosti od počátku. Bod se pohybuje po elipse o rovnici $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ a to od bodu $(a,0)$ k bodu $(0,b)$.



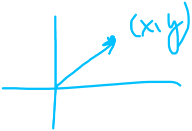
a) $\|\vec{F}\| = \sqrt{x^2+y^2}$

a $\boxed{\vec{F} = (-x,-y)}$

b) $\|\vec{F}\| = \frac{1}{\sqrt{x^2+y^2}}$

$\vec{F} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{(-x,-y)}{\sqrt{x^2+y^2}}$

b $\boxed{\vec{F} = \frac{(-x,-y)}{x^2+y^2}}$



$\int_{(c)} \vec{F}\, d\vec{s}$

parametrizace (C)

$\varphi(t) = (\overset{x}{a\cos t}, \overset{y}{b\sin t})$

$t \in \langle 0, \frac{\pi}{2} \rangle$

$\vec{F} = \lambda(-x,-y)$

$\|\vec{F}\| = \lambda\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}$, $\lambda = \frac{1}{x^2+y^2}$

a) $\displaystyle\int_{(c)} \vec{F}\,d\vec{s} = \int_0^{\frac{\pi}{2}} \underbrace{(-a\cos t, -b\sin t)}_{\vec{F}(\varphi)} \cdot \underbrace{(-a\sin t, b\cos t)}_{\varphi'}\,dt$

$\displaystyle = \int_0^{\frac{\pi}{2}} (a^2 \underbrace{\cos t \sin t} - b^2 \underbrace{\sin t \cos t})\,dt = (a^2-b^2)\int_0^{\frac{\pi}{2}} \overbrace{\cos t \sin t}^{\frac{1}{2}\sin 2t}\,dt$

$\displaystyle = \left| \begin{array}{l} \cos t = u \\ -\sin t\,dt = du \\ t=0,\ u=1 \\ t=\frac{\pi}{2}\ u=0 \end{array} \right| = (a^2-b^2)\int_1^0 -u\,du = (a^2-b^2)\int_0^1 u\,du$

$\displaystyle = (a^2-b^2)\left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}(a^2-b^2)$

b) $\displaystyle\int_{(c)} \vec{F}\,d\vec{s} = \int_0^{\frac{\pi}{2}} \frac{(a^2-b^2)\cos t \sin t}{a^2\cos^2 t + \underbrace{b^2\sin^2 t}_{b^2(1-\cos^2 t)}}\,dt = \int_0^{\frac{\pi}{2}} \frac{(a^2-b^2)\cos t\sin t}{(a^2-b^2)\cos^2 t + b^2}\,dt$

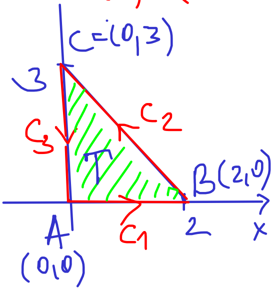
$\displaystyle = \left| \begin{array}{l} u = (a^2-b^2)\cos^2 t + b^2 \\ du = \underbrace{(a^2-b^2)}\,2\,\underbrace{\cos t}\cdot\underbrace{(-\sin t)}\,\underbrace{dt} \end{array} \right| = -\frac{1}{2}\int_{a^2}^{b^2} \frac{du}{u} = -\frac{1}{2}\left[\ln u\right]_{a^2}^{b^2}$

$t=0, \quad u=a^2$
$t=\frac{1}{2}, \quad u=b^2$

$$= -\frac{1}{2}\left(\ln b^2 - \ln a^2\right) = -\frac{1}{2}\ln\frac{b^2}{a^2} = -\ln\frac{b}{a} = \ln\frac{a}{b}$$

$$\int_{(C)} \vec{F}\, d\vec{s} = \int_{(C)} F_1\, dx + F_2\, dy \qquad \vec{F} = (F_1, F_2)$$

Vypočítejte $\boxed{\int_{(C)} xdy}$, kde (C) je kladně orientovaný obvod trojúhelníku tvořený osami souřadnic a přímkou o rovnici $\frac{x}{2} + \frac{y}{3} = 1$. $(C) = (\partial T)$



$$\vec{F} = (0, x)$$

$$\int_{(C)} \vec{F}\, d\vec{s} = \int_{(C_1)} \vec{F}\, d\vec{s} + \int_{(C_2)} \vec{F}\, d\vec{s} + \int_{(C_3)} \vec{F}\, d\vec{s} = 3$$

C_1 $\quad \varphi(t) = A + (B-A)t\,, \quad t \in \langle 0, 1\rangle$

$$= (0,0) + (2,0)t = (\overbrace{2t}^{x}, \overbrace{0}^{y})$$

$$\int_{(C_1)} \vec{F}\, d\vec{s} = \int_0^1 \underbrace{(0, 2t)}_{\vec{F}(\varphi)} \cdot \underbrace{(2,0)}_{\varphi'}\, dt = 0$$

C_2 $\quad \varphi(t) = B + (C-B)t = (2,0) + (-2,3)t = (\overbrace{2-2t}^{x}, 3t) \quad t \in \langle 0, 1\rangle$

$$\int_{(C_2)} \vec{F}\, d\vec{s} = \int_0^1 (0, 2-2t) \cdot (-2, 3)\, dt = 6\int_0^1 (1-t)\, dt =$$

$$= 6\left[t - \frac{t^2}{2}\right]_0^1 = 3$$

C_3 $\varphi(t) = C + (A-C)t = (0,3) + (0,-3)t = (\overset{x}{\widehat{0}}, 3-3t)$

$t \in \langle 0,1 \rangle$

$$\int_{(C_3)} \vec{F}\, d\vec{s} = \int_0^1 \underbrace{(0,0)}_{F=(0,x)} \cdot (0,-3)\, dt = 0$$

Věta (Green)

$$\int_{(\partial T)} \vec{F}\, d\vec{s} = \iint_T \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \iint_T 1 = \frac{3 \cdot 2}{2}$$

$$\vec{F} = (0,x), \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$$

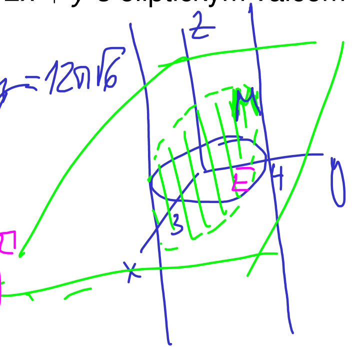
$$\rightarrow \iint_M f\,dS = \iint_A f(\Phi)\cdot\left\|\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}\right\|$$

parametrizace M

$\Phi(u,v): A \to M$, $A \subseteq \mathbb{R}^2$

Stanovte obsah průniku roviny o rovnici $z = 2x + y$ s eliptickým válcem daným nerovnicí $\frac{x^2}{9} + \frac{y^2}{16} \leq 1$.

$$\iint_M 1\,dS = \iint_E \sqrt{6} = \sqrt{6}\cdot \underbrace{\text{obsah elipsy}}_{\pi\cdot a\cdot b} = 12\pi\sqrt{6}$$



$$M = \{(x,y,z) \mid (x,y)\in E,\ z = g(x,y)\}$$

$$\rightarrow \iint_M f\,dS = \iint_E f(x,y,g(x,y))\cdot\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}$$

$g(x,y) = 2x+y \qquad \sqrt{1+2^2+1^2} = \sqrt{6}$

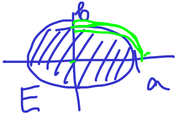
parametrizace $\Phi(x,y) = (x, y, 2x+y)$, $\quad \frac{\partial\Phi}{\partial x} = (1,0,2)$

$$\frac{\partial\Phi}{\partial x}\times\frac{\partial\Phi}{\partial y} = \begin{vmatrix} 1 & 0 & e_1 \\ 0 & 1 & e_2 \\ 2 & 1 & e_3 \end{vmatrix} = e_1(-2) - e_2\cdot 1 + e_3\cdot 1 = (-2,-1,1)$$

$\frac{\partial\Phi}{\partial y} = (0,1,1)$

$$\left\| \frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} \right\| = \sqrt{4+1+1} = \sqrt{6}$$

$$\iint_E 1 = \int_0^{2\pi} \int_0^1 ab\rho \, d\rho \, d\varphi = ab \int_0^{2\pi} \left[\frac{\rho^2}{2} \right]_0^1 d\varphi = ab \, \frac{1}{2} \cdot 2\pi = \pi ab$$



Substituce

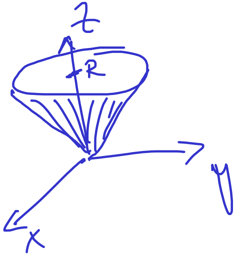
$x = a\rho \cos\varphi \qquad \rho \in \langle 0, 1 \rangle$

$y = b\rho \sin\varphi \qquad \varphi \in \langle 0, 2\pi \rangle$

$$J = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} a\cos\varphi & -a\rho\sin\varphi \\ b\sin\varphi & b\rho\cos\varphi \end{pmatrix}$$

$$\Delta_\Phi = ab\rho\cos^2\varphi + ab\rho\sin^2\varphi = ab\rho$$

Vypočtěte plochu pláště kužele $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq R$.



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