Matematická analýza 2

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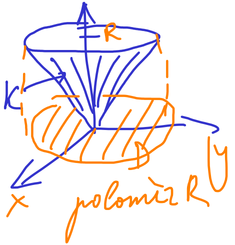
$K = \{(x,y,z),\ (x,y)\in D,\ z = \underbrace{\sqrt{x^2+y^2}}_{g(x,y)}\}$

Vypočtěte plochu pláště kuželе $z = \sqrt{x^2+y^2}$, $0 \le z \le R$.

$$\iint_K \overset{f}{1}\, dS = \iint_D \overset{f(\Phi)}{}\left\| \frac{\partial \Phi}{\partial \rho} \times \frac{\partial \Phi}{\partial \varphi} \right\| d\rho\, d\varphi$$



parametrizace $\Phi(\rho,\varphi)$ pláště kužele

$$= \iint_D \overset{1}{\underset{f(x,y,\sqrt{x^2+y^2})}{}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} =$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}2x = \frac{x}{\sqrt{x^2+y^2}},\quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$= \iint_D \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}} = \iint_D \sqrt{2} = \sqrt{2}\iint_D 1 = \sqrt{2}\,\pi R^2$$

Plošný integrál funkci f přes plochu M je

1) $$\iint_M f\,dS = \iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}$$

pro $M=\{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in D,\ z=g(x,y)\}$

1

2) $$\iint_M f\,dS = \iint_D f(\varphi_1,\varphi_2,\varphi_3)\cdot\left\|\frac{\partial\varphi}{\partial u}\times\frac{\partial\varphi}{\partial v}\right\|$$

pro parametrizaci $\varphi: D\to M$, $u,v\in D$

$$\varphi(u,v)=(\varphi_1(u,v),\ \varphi_2(u,v),\ \varphi_3(u,v))$$

obsah plochy M je $\iint_M 1\,dS$

Plášť kužele

1) buď $K=\{(x,y,z) \mid (x,y)\in D,\ z=\sqrt{x^2+y^2}\}$, D je kruh o poloměru R

2) nebo $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $z=\rho$

$\Phi(\rho,\varphi)=(\rho\cos\varphi,\ \rho\sin\varphi,\ \rho)$, $\rho\in\langle 0,R\rangle$, $\varphi\in\langle 0,2\pi\rangle$

$z=\sqrt{x^2+y^2}=\sqrt{\rho^2\cos^2\varphi+\rho^2\sin^2\varphi}=\rho$

1) $\iint_K 1\,dS = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \iint_D \sqrt{2} = \sqrt{2}\underbrace{\iint_D 1}_{\text{obsah kruhu}} = \sqrt{2}\pi R^2$

2

2) $\frac{\partial \Phi}{\partial \rho} = (\cos\varphi, \sin\varphi, 1)$

$\frac{\partial \Phi}{\partial \varphi} = (-\rho\sin\varphi, \rho\cos\varphi, 0)$

jsou kolmé

$\left\|\frac{\partial \Phi}{\partial \rho} \times \frac{\partial \Phi}{\partial \varphi}\right\| = \left\|\frac{\partial \Phi}{\partial \rho}\right\| \cdot \left\|\frac{\partial \Phi}{\partial \varphi}\right\| = \sqrt{\cos^2\varphi+\sin^2\varphi+1}\cdot \rho\sqrt{\cos^2\varphi+\sin^2\varphi}$

$= \sqrt{2}\cdot\rho$

$\iint_K 1\,dS = \int_0^R \underbrace{\int_0^{2\pi} \sqrt{2}\rho \,d\varphi}\, d\rho = \sqrt{2}\cdot 2\pi \int_0^R \rho\, d\rho = 2\sqrt{2}\pi\left[\frac{\rho^2}{2}\right]_0^R =$

$= \sqrt{2}\pi R^2$

$$x^2+y^2+z^2=a^2 \qquad z=g(x,y)$$

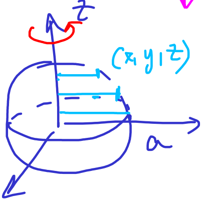
Vypočtěte moment setrvačnosti povrchu kouli o poloměru a se středem v počátku vzhledem k ose z. Hustota je konstantní funkce $\rho = 1$.

$$I = \iint_{\text{sfera}} r^2 \rho$$

r je vzdálenost od osy rotace

$$r^2 = x^2+y^2$$

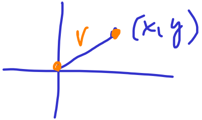
$$r=\sqrt{x^2+y^2}, \quad \rho = 1$$



parametrizace sfery

$$x = a\sin\theta\cos\varphi$$
$$y = a\sin\theta\sin\varphi$$
$$z = a\cos\theta$$

$$\varphi \in \langle 0, 2\pi\rangle$$
$$\theta \in \langle 0, \pi\rangle$$



$$r^2(\Phi) = a^2\sin^2\theta$$

$$\Phi(\varphi,\theta) = (a\sin\theta\cos\varphi,\ a\sin\theta\sin\varphi,\ a\cos\theta)$$

$$\left\|\frac{\partial\Phi}{\partial\varphi}\times\frac{\partial\Phi}{\partial\theta}\right\| = \left\|\frac{\partial\Phi}{\partial\varphi}\right\|\cdot\left\|\frac{\partial\Phi}{\partial\theta}\right\|\cdot\underbrace{\sin\alpha}_{\text{pro } \alpha=\frac{\pi}{2} \text{ je to } 1} = a^2\sin\theta$$

$$\frac{\partial \varphi}{\partial \varphi} = (-a \sin\theta \sin\varphi,\ a \sin\theta \cos\varphi,\ 0)$$

$$\frac{\partial \varphi}{\partial \varphi} \perp \frac{\partial \varphi}{\partial \theta}$$

$$\frac{\partial \varphi}{\partial \theta} = (a \cos\theta \cos\varphi,\ a \cos\theta \sin\varphi,\ -a \sin\theta)$$

$$\cancel{-a^2 \sin\theta\cos\theta \sin\varphi\cos\varphi} + \cancel{a^2 \sin\theta\cos\theta \sin\varphi\cos\varphi} + 0$$

$$\|(x,y,z)\| = \sqrt{x^2+y^2+z^2}$$

$$\left\|\frac{\partial \varphi}{\partial \varphi}\right\| = a\sqrt{\sin^2\theta \sin^2\varphi + \sin^2\theta \cos^2\varphi} = a \cdot \sqrt{\sin^2\theta}$$

$$= a \sin\theta$$

bo $\theta \in \langle 0, \pi \rangle$

$$\left\|\frac{\partial \varphi}{\partial \theta}\right\| = a\sqrt{\cos^2\theta + \sin^2\theta} = a$$

$$I = \iint_{\text{sfera}} v^2 \rho \, ds = \int_0^{\pi}\int_0^{2\pi} a^2 \sin^2\theta \cdot a^2 \sin\theta \, d\varphi \, d\theta$$

$$= a^4 2\pi \int_0^\pi \sin^3\theta \, d\theta = 2\pi a^4 \int_0^\pi (1-\cos^2\theta) \underbrace{\sin\theta \, d\theta}$$

$$= \left| \begin{array}{l} u = \cos\theta \\ du = -\underbrace{\sin\theta \, d\theta} \end{array} \right| = 2\pi a^4 \int_{-1}^{\pm 1} (1-u^2)(\pm du) =$$

$\theta = 0, \quad u = 1$

$\theta = \pi, \quad u = -1$

$$= 2\pi a^4 \left[u - \frac{u^3}{3} \right]_{-1}^{1} = \frac{8}{3}\pi a^4$$

$$1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right) = 2 - \frac{2}{3} = \frac{4}{3}$$

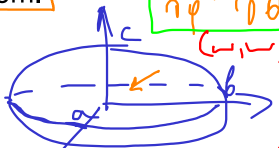
$\vec{F} = (F_1, F_2, F_3)$ $\quad \iint_{(M)} \vec{F}\, d\vec{S} = \iint_{(M)} F_1\, dy\, dz + F_2\, dx\, dz + F_3\, dx\, dy$

Vypočtěte $\iint_{(M)} z\,dx\,dy$, kde M je povrch elipsoidu $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$, orientovaný vnějším normálovým polem.

$\boxed{\vec{F} = (0, 0, z)}$ param $\Phi(\varphi, \theta)$

$\frac{\partial \Phi}{\partial \varphi} \times \frac{\partial \Phi}{\partial \theta}$ $(\_, \_, \_)$ <0

$\iint_{(M)} \vec{F}\, d\vec{S} = \int_0^{\pi}\int_0^{2\pi} \vec{F}(\Phi) \cdot \left(\frac{\partial \Phi}{\partial \varphi} \times \frac{\partial \Phi}{\partial \theta}\right) d\varphi\, d\theta$



I. $x>0, y>0, z>0$

$\Phi(\varphi, \theta) = (a \sin\theta \cos\varphi,\ b \sin\theta \sin\varphi,\ c \cos\theta)$

e_1 0 $\quad e_2$ 0 ! $\quad e_3$ $\cos\theta$

$\frac{\partial \Phi}{\partial \varphi} = (-a \sin\theta \sin\varphi,\ b \sin\theta \cos\varphi,\ 0)$

$\frac{\partial \Phi}{\partial \theta} = (a \cos\theta \cos\varphi,\ b \cos\theta \sin\varphi,\ -c \sin\theta)$

II | I
III | IV
kvadranty

$-\int_0^{\pi}\int_0^{2\pi} c \cos\theta \cdot (-ab \sin\theta \cos\theta \sin^2\varphi - ab \sin\theta \cos\theta \cos^2\varphi)\, d\varphi\, d\theta$

$-ab \sin\theta \cos\theta$

$$= \int_0^{\pi} \underbrace{\int_0^{2\pi} abc \cos^2\theta \sin\theta \, d\varphi}\, d\theta =$$

$$= abc\, 2\pi \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta = \left| \begin{matrix} u = \cos\theta \\ du = -\sin\theta \, d\theta \end{matrix} \right|$$

$$= 2\pi abc \int_1^{-1} u^2 (-du) = 2\pi abc \left[\frac{u^3}{3} \right]_{-1}^{1}$$

$$\frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

$$= \frac{4}{3} \pi abc$$

$$\iint_{\substack{\text{povrch} \\ \text{elipsoidu}}} \vec{F}\, dS = \iiint_{\text{elipsoid}} \operatorname{div} \vec{F} = \overbrace{\iiint_{\text{elipsoid}} 1}^{\text{objem elipsoidu}}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

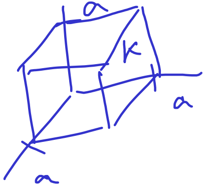
$$\operatorname{div} \vec{F} = 1$$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad , \quad \vec{F} = (0, 0, z)$$

$$\operatorname{div}\vec{F} = 2x+2y+2z$$

Pomocí Gaussovy věty vypočítejte tok pole $\vec{F}(x,y,z) = (x^2, y^2, z^2)$ přes povrch krychle $\langle 0, a\rangle^3$, orientovaný vnějším normálovým polem.

$$\iint_{\partial K} \vec{F}\, d\vec{S} = \iiint_K \operatorname{div}\vec{F} = \int_0^a\int_0^a\int_0^a (2x+2y+2z)\,dx\,dy\,dz$$



$$= \int_0^a\int_0^a \left[x^2+2yx+2zx\right]_0^a dy\,dz = \int_0^a\int_0^a (a^2+2ya+2za)\,dy\,dz$$

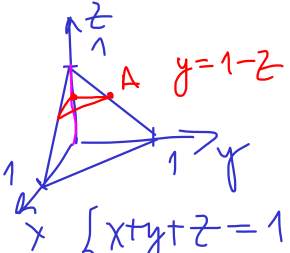
$$= \int_0^a \left[a^2y+y^2a+2zay\right]_0^a dz = \int_0^a (a^3+a^3+2za^2)\,dz$$

$$= \left[2a^3z+z^2a^2\right]_0^a = 2a^4+a^4 = 3a^4$$

$\vec{F} = (xz, xy, yz)$, $\operatorname{div} \vec{F} = z + x + y$

Pomocí Gaussovy věty vypočítejte $\iint_{(M)} \overbrace{xz}^{F_1}\,dydz + \overbrace{xy}^{F_2}\,dzdx + \overbrace{yz}^{F_3}\,dxdy$, kde M je povrch jehlanu omezeného rovinami $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$. Orientace je dána vnějším normálovým polem.

$$\underset{\text{povrch jehlanu}}{\iint} \vec{F}\,d\vec{S} = \underset{\text{jehlan}}{\iiint} \operatorname{div} \vec{F}$$



$$\begin{cases} x+y+z = 1 \\ x = 0 \end{cases}$$

$y + z = 1$

$y = 1 - z$

$$= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} (x+y+z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-z} \left[\frac{x^2}{2} + (y+z)x \right]_0^{1-y-z} dy\,dz$$

$$= \int_0^1 \int_0^{1-z} \frac{(1-y-z)^2}{2} + (y+z)(1-y-z)\,dy\,dz$$

$$= \int_0^1 \int_0^{1-z} (1-y-z)\left(\frac{1-y-z}{2} + y+z\right) dy\,dz$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-z} (1\overbrace{-y-z}^{-a})(1+\overbrace{y+z}^{a})\, dy\,dz$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-z} \left(1-(y+z)^2\right) dy\,dz =$$

$$= \frac{1}{2}\int_0^1 \left[y - \frac{(y+z)^3}{3}\right]_0^{1-z} dz = \frac{1}{2}\int_0^1 1-z - \left(\frac{1}{3} - \frac{z^3}{3}\right) dz$$

$$= \frac{1}{2}\left[\frac{2}{3}z - \frac{z^2}{2} + \frac{z^4}{12}\right]_0^1 = \frac{1}{2}\left(\frac{2}{3} - \frac{1}{2} + \frac{1}{12}\right)$$

$$= \frac{1}{2}\,\frac{8-6+1}{12} = \frac{1}{8}$$

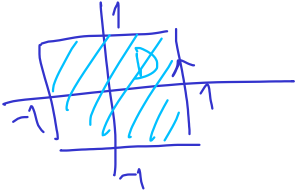
$$F(x,y) = (y^2, x)$$

Pomocí Greenovy věty vypočtěte $\int_{(C)} y^2 dx + x dy$, kde C je kladně orientovaná hranice čtverce $\langle -1, 1\rangle^2$.

$C = \partial D$

$$\int_{(\partial D)} \vec{F} d\vec{s} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= \int_{-1}^{1}\int_{-1}^{1} (1-2y)\, dx dy$$



DÚ

Vypočtěte $\iint_{(M)} rot\vec{F}d\vec{S}$, kde $\vec{F}(x,y,z) = (y,z,x)$, a M je část paraboloidu $z = 1 - x^2 - y^2$, $z \geq 0$. Normálové pole určující orientaci má nezápornou z-tovou složku.

DÚ

Určete potenciály následujících vektorových polí v uvedených oblastech (existují-li)

1) $\vec{F} = (-y^2 - 2xz, 2yz - 2xy, y^2 - x^2)$ v $\mathbb{R}^3$;

2) $\vec{F} = (y^2 \cos x, 2y \sin x)$ v $\mathbb{R}^2$.