Matematická analýza 2

Natalie Žukovec

14. cvičení 2021

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \qquad a_n \in \mathbb{R}$$
x_0 je střed

$$(5x)^n = 5^n \cdot x^n$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}, \quad x_0 = 0$$

e^{5x}, $x_0 = 0$

$$e^{5x} = \left| t = 5x \right| = e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{(5x)^n}{n!} = \sum_{n=0}^{\infty} \frac{5^n}{n!} x^n$$

e^x, $x_0 = 2$

$$e^x = \sum_{n=0}^{\infty} \underbrace{\;?\;} (x-2)^n$$

$$e^x = e^{(x-2)+2} = \left| t = x-2 \right| = e^{t+2} = e^2 e^t = e^2 \sum_{n=0}^{\infty} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n$$

5^x, $x_0 = 0$

$$5^x = e^{\ln 5^x} = e^{x \ln 5} = \left| t = x \cdot \ln 5 \right| = \sum_{n=0}^{\infty} \frac{(x \cdot \ln 5)^n}{n!} = \sum_{n=0}^{\infty} \frac{\ln^n 5}{n!} x^n$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n,\ |x|<1$$

$$\frac{1}{2-x},\ x_0=0$$

$$\frac{1}{2-x}=\frac{1}{2\left(1-\frac{x}{2}\right)}=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}x^n$$

$$\frac{1}{2-x},\ x_0=-3$$

$$\sum_{n=0}^{\infty}\ \ \ (x+3)^n$$

$$\frac{1}{2-x}=\frac{1}{5-(x+3)}=\frac{1}{5}\ \frac{1}{1-\frac{x+3}{5}}=\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x+3}{5}\right)^n=$$

$$=\sum_{n=0}^{\infty}\frac{1}{5^{n+1}}(x+3)^n$$

$$\frac{1}{1-x}$$

Nalezěte Taylorův rozvoj funkce $f(x) = \dfrac{1-3x}{3+2x-x^2}$ v bodě $x_0 = 0$.

$$f(x) = \frac{3x-1}{x^2-2x-3} = \frac{3x-1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} = \boxed{\frac{2}{x-3} + \frac{1}{x+1}}$$

$$D = 4+12 = 16$$

$$x_{1,2} = \frac{2 \pm 4}{2} \begin{cases} 3 \\ -1 \end{cases}$$

$$3x-1 = A(x+1) + B(x-3)$$

$$x!\quad 3 = A + B \qquad 4 = 4B$$

$$x^0!\quad -1 = A - 3B \qquad B = 1$$

$$A = 2$$

$$A = \frac{3x-1}{x+1}\Big|_{x=3} = \frac{8}{4} = 2$$

$$= \sum_{n=0}^{\infty} \left(\frac{-2}{3^{n+1}} + (-1)^n \right) x^n$$

$$B = \frac{3x-1}{x-3}\Big|_{x=-1} = \frac{-4}{-4} = 1$$

$$f(x) = \frac{2}{-3} \frac{1}{1-\frac{x}{3}} + \frac{1}{1-(-x)} = \frac{-2}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n + \sum_{n=0}^{\infty} (-x)^n =$$

Vyšetřete, pro která $x \in \mathbb{R}$ konverguje řada

$$\sum_{n=1}^{\infty}\left(1+\frac{2}{n}\right)x^n,$$

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$\frac{1}{R} = \limsup_{n\to\infty}\sqrt[n]{|a_n|}$$

a pomocí derivování nebo integrace nalezněte její součet.

$$\frac{1}{R} = \lim_{n\to\infty}\frac{1+\frac{2}{n+1}}{1+\frac{2}{n}} = 1, \quad R=1, \quad x \in (-1,1)$$

$$\sum_{n=1}^{\infty}\left(1+\frac{2}{n}\right)x^n = \sum_{n=1}^{\infty} x^n + 2\underbrace{\sum_{n=1}^{\infty}\frac{x^n}{n}} = \frac{x}{1-x} - 2\ln(1-x)$$

$$\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)' = \sum_{n=1}^{\infty}\left(\frac{x^n}{n}\right)' = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

$$\int_0^x \frac{1}{1-t}$$

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = -\ln(1-x) + \cancel{c}, \quad \text{pro } x=0 \quad 0 = \underbrace{-\ln 1}_{0} + c$$

Určete poloměr konvergence a sečtěte mocninnou řadu na vnitřku oboru konvergence

$$\sum_{n=0}^{\infty}(n+(-2)^n)x^n$$

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{n+1+(-2)^{n+1}}{n+(-2)^n}\right|=\lim_{n\to\infty}\left|\frac{\frac{n+1}{(-2)^n}+(-2)}{\frac{n}{(-2)^n}+1}\right|=2,\quad R=\frac{1}{2}$$

$$\frac{n+1}{(-2)^n}\to 0,\qquad \frac{n}{(-2)^n}\to 0$$

$$x\in\left(-\frac{1}{2},\frac{1}{2}\right)$$

$$\sum_{n=0}^{\infty}(n+(-2)^n)x^n=\sum_{n=0}^{\infty}nx^n+\sum_{n=0}^{\infty}(-2)^n x^n=\frac{x}{(1-x)^2}+\frac{1}{1+2x}$$

$$\sum_{n=1}^{\infty}nx^n = x\sum_{n=1}^{\infty}nx^{n-1}$$

$$\sum_{n=0}^{\infty}(-2)^n x^n=\sum_{n=0}^{\infty}(-2x)^n=\frac{1}{1-(-2x)}=\frac{1}{1+2x}$$

$$\left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=\cancel{0}\,1}^{\infty} n x^{n-1} = \left(\frac{1}{1-x}\right)' = \left((1-x)^{-1}\right)' =$$

$$= (-1)(1-x)^{-2}\cdot(-1) = \frac{1}{(1-x)^2}$$

Nechť f je T-periodická funkce a existuje $\int_0^T |f(x)|\,dx$.
Jeji Fourierovou řadu definujeme jako

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi n}{T}x\right)+b_n\sin\left(\frac{2\pi n}{T}x\right)\right],$$

pro $T=2\pi$

$\cos nx$

$\frac{2}{T}=\frac{1}{\pi}$

kde

$f(x)$ je sudá

$b_n=0$

pro $T=\pi$

$\frac{2}{T}=\frac{2}{\pi}$

$\cos(2nx)$

$$a_0=\frac{2}{T}\int_0^T f(x)\,dx,$$

$$a_n=\frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi n}{T}x\right)dx,$$

$$b_n=\frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi n}{T}x\right)dx.$$

$\int_0^T$ nebo $\int_{-\frac{T}{2}}^{\frac{T}{2}}$

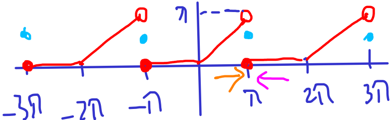
Najděte Fourierův rozvoj 2π-periodického rozšíření funkce s daným předpisem na intervalu $\langle -\pi, \pi)$ a vyšetřete konvergenci

$$\frac{1}{2}(\pi + 0) = \frac{\pi}{2}$$

$$f(x) = \begin{cases} 0, & x \in \langle -\pi, 0) \\ x, & x \in \langle 0, \pi) \end{cases}$$

$$\frac{1}{2}\left(f(x-) + f(x+)\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$



$$f(\pi-) = \pi$$
$$f(\pi+) = 0$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x \cos nx\,dx = \left| \begin{matrix} u = x & v' = \cos nx \\ u' = 1 & v = \frac{\sin nx}{n} \end{matrix} \right| =$$

$n = 1, 2, 3, \ldots$

$$= \frac{1}{\pi}\cancel{\left[x\frac{\sin nx}{n}\right]_0^{\pi}} - \frac{1}{\pi}\int_0^{\pi}\frac{\sin nx}{n}\,dx = -\frac{1}{\pi}\left[-\frac{\cos nx}{n^2}\right]_0^{\pi} =$$

$$= \frac{1}{\pi n^2}\left((-1)^n - 1\right)$$

$$\cos n\pi = (-1)^n$$

$$b_n = \frac{1}{\pi}\int_0^\pi x \sin nx \, dx = \frac{1}{\pi}\left[x \, \frac{-\cos nx}{n}\right]_0^\pi + \frac{1}{\pi}\int_0^\pi \frac{+\cos nx}{n}\, dx$$

$$= \frac{1}{\pi}\pi \cdot \frac{-(-1)^n}{n} - \cancel{\frac{1}{\pi}\left[\frac{\sin nx}{n^2}\right]_0^\pi} = \frac{(-1)^{n+1}}{n}$$

$$f(x) \sim \underbrace{\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right)}$$

součet řady je $\begin{cases} f(x) & \text{pro } x \neq (2k+1)\pi, \; k \in \mathbb{Z} \\ \frac{\pi}{2} & \text{pro } x = (2k+1)\pi \end{cases}$

v bodě 0

$$f(0) = 0 = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2} = \boxed{\frac{\pi}{4} + \sum_{k=0}^{\infty} \frac{-1-1}{\pi(2k+1)^2} = 0}$$

$n = 2k+1$

$$\underbrace{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots}=\frac{\pi^2}{8}$$

$$\frac{\pi}{4}=\sum_{k=0}^{\infty}\frac{2}{\pi(2k+1)^2}$$

$$\frac{\pi}{4}=\frac{2}{\pi}\underbrace{\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}}$$

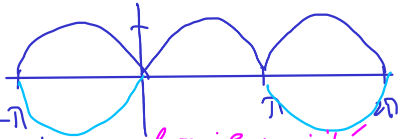
$$\frac{\pi^2}{8}=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)$$

Nalezněte Fourierovou řadu pro periodické rozšíření funkce $f(x) = \sin x$, $x \in \langle 0, \pi)$ a určete její součet.

$T = \pi$

$f(x) = f(-x)$ je sudá



f-ce je spojitá

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin x\,dx = \frac{2}{\pi}\left[-\cos x\right]_0^\pi = \frac{4}{\pi}$$

$-(-1)-(-1) = 2$

$$a_n = \frac{2}{\pi}\int_0^\pi \underbrace{\sin x}_{\alpha}\,\underbrace{\cos 2nx}_{\beta}\,dx = \frac{2}{\pi}\,\frac{1}{2}\int_0^\pi \sin(2n+1)x - \sin(2n-1)x\,dx$$

nebo per partes

$\alpha+\beta = (2n+1)x$

$\alpha-\beta = (1-2n)x$

$-(2n-1)x$

$-1-1$

$$= \frac{1}{\pi}\left[\frac{-\cos(2n+1)x}{2n+1} + \frac{\cos(2n-1)x}{2n-1}\right]_0^\pi =$$

$$\cos(2n+1)\pi = \cos(2n\pi+\pi) = \cos\pi = -1$$
$$\cos(2n-1)\pi = \cos(2n\pi-\pi) = \cos(-\pi) = -1$$

$$= \frac{1}{\pi}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right) = \frac{-4}{\pi(4n^2-1)}$$

$$f(x) \sim \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2-1)} \cos(2nx) = f(x)$$

$|\sin x|$ na $\mathbb{R}$

per partes

$$A = \int_0^{\pi} \sin x \cos 2nx \, dx = \left| \begin{array}{ll} u = \sin x & v' = \cos 2nx \\ u' = \cos x & v = \frac{\sin 2nx}{2n} \end{array} \right| =$$

$$= \underbrace{\left[\frac{\sin x \sin 2nx}{2n}\right]_0^{\pi}}_{=0} - \frac{1}{2n}\int_0^{\pi} \cos x \sin 2nx \, dx = \left| \begin{array}{ll} u = \cos x & v' = \sin 2nx \\ u' = -\sin x & v = \frac{-\cos 2nx}{2n} \end{array} \right|$$

$$= -\frac{1}{2n}\left[\frac{-\cos x \cos 2nx}{2n}\right]_0^{\pi} + \frac{1}{4n^2}\int_0^{\pi} \sin x \cos 2nx \, dx = \frac{1}{4n^2}(-1-1) + \frac{1}{4n^2} A$$

$$A = \frac{-2}{4n^2-1}$$

Najděte Fourierův rozvoj 2π-periodického rozšíření funkce a vyšetřete konvergenci: $f(x) = x^2$ na intervalu $\langle -\pi, \pi)$