Matematická analýza 2

Natalie Žukovec

3. cvičení 2021

Směrová derivace f v bodě x_0 ve směru vektoru h

$f: G \to \mathbb{R}$, $G \subseteq \mathbb{R}^n$, otevřená, $x_0 \in G$, $h \in \mathbb{R}^n$

$$\frac{\partial f}{\partial h}(x_0) = \lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t}$$

$\frac{\partial f}{\partial x}$ je směrová derivace ve směru $(1,0)$

Spočtěte parciální derivace funkce

$Df = \mathbb{R}^2$

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

$$\frac{\partial f}{\partial x} = \frac{y\sqrt{x^2+y^2} - xy\,\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y(x^2+y^2) - x^2 y}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{y^3}{(x^2+y^2)^{3/2}}$$

$(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial y} = \frac{x^3}{(x^2+y^2)^{3/2}}$$

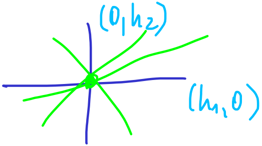
$x_0 = (0,0)$, $h = (1,0)$

$x_0 + th = (0,0) + t(1,0) = (t,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{\overset{=0}{f(t,0)} - \overset{=0}{f(0,0)}}{t} = 0$$

$(x,y) = (0,0)$

$f(x,0) = 0$



$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) \neq 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$$

1. Je funkce f v bodě $(0,0)$ spojitá? ANO
2. Jsou funkce $\frac{\partial f}{\partial x}(x,y)$ a $\frac{\partial f}{\partial y}(x,y)$ v bodě $(0,0)$ spojité? NE
3. Najděte všechny směrové derivace v bodě $(0,0)$. existují jenom $\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)$
4. Je funkce f v bodě $(0,0)$ diferencovatelná?

$x_0=(0,0)$, $h=(h_1,h_2)$, $x_0+th=(0,0)+t(h_1,h_2)=(th_1,th_2)$

$$\frac{\partial f}{\partial h}(0,0) = \lim_{t\to 0} \frac{f(th_1,th_2)-\overbrace{f(0,0)}^{0}}{t} = \lim_{t\to 0} \frac{1}{t}\cdot\frac{t h_1 t h_2}{\sqrt{t^2h_1^2+t^2h_2^2}}$$

$$= \frac{h_1h_2}{\sqrt{h_1^2+h_2^2}} \boxed{\lim_{t\to 0}\frac{t}{\sqrt{t^2}}}$$ neex

$\frac{h_1h_2}{\sqrt{h_1^2+h_2^2}} \neq 0$

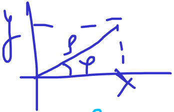
$\sqrt{t^2}=|t|$

$$\lim_{t\to 0^+}\frac{t}{t}=1 \qquad \lim_{t\to 0^-}\frac{t}{-t}=-1$$

$$\frac{xy}{\sqrt{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{\rho\to 0} \frac{\rho\cos\varphi\,\rho\sin\varphi}{\sqrt{\rho^2\cos^2\varphi+\rho^2\sin^2\varphi}} =$$

$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$



$$= \lim_{\rho\to 0} \frac{\rho^2\cos\varphi\sin\varphi}{\rho}$$

$$\rho^2(\cos^2\varphi+\sin^2\varphi) = \rho^2$$

$\sqrt{\rho^2} = \rho$, protože $\rho \geq 0$

$$= \lim_{\rho\to 0} \rho \underbrace{\cos\varphi\sin\varphi}_{\leq 1}$$

$$= /0 \cdot \text{omez}/ = 0$$

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y^3}{(x^2+y^2)^{3/2}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

$$(x^2)^{3/2} = x^3$$

$$\lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{3/2}}$$

neex

$\boxed{\text{po } x=y}$

$$\lim_{\substack{x\to 0\\ y=x}} \frac{x^3}{(2x^2)^{3/2}} = \boxed{\frac{1}{2^{3/2}}}$$

pro $\boxed{y=0}$ $\lim_{\substack{x \to 0 \\ y=0}} \frac{0}{(x^2)^{3/2}} = \boxed{0}$

$$\boxed{\frac{\partial f}{\partial h}(x,y) = h \cdot \operatorname{grad} f(x,y)}$$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ jsou spojité v (x,y)

$$\boxed{\operatorname{grad} f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)}$$

$D(f) = \mathbb{R}^3 \setminus \{(0,0,0)\}$

Spočtěte parciální derivace funkce $f(x, y, z) = \ln(\overbrace{x^2 + y^2 + z^4}^{>0})$.

$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2+z^4}$, $\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2+z^4}$, $\frac{\partial f}{\partial z} = \frac{4z^3}{x^2+y^2+z^4}$ v bodě $(1,-1,0)$

Pro funkci f a vektory $u_1 = (1, -2, 1)$, $u_2 = (0, 1, 1)$, $u_3 = (1, 0, 0)$ vypočtěte derivace

$$\frac{\partial f}{\partial u_1}(x, y, z) = (1,-2,1)\cdot \operatorname{grad} f(x,y,z) = \frac{2x-4y+4z^3}{x^2+y^2+z^4}$$

$$\frac{\partial f}{\partial u_2}(x, y, z) = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = \frac{2y+4z^3}{x^2+y^2+z^4}$$

$$\frac{\partial f}{\partial u_3}(1, 1, 1) = \frac{\partial f}{\partial x}(1,1,1) = \frac{2}{3}$$

$$\frac{\partial f}{\partial (u_1+u_2)}(1, -1, 0) = (1,-1,2)\cdot \operatorname{grad} f(1,-1,0) = 2$$

$u_1+u_2 = (1,-1,2)$

$\left(\frac{2}{2}, \frac{-2}{2}, \frac{0}{2}\right) = (1,-1,0)$

Určete derivaci funkce $f(x, y) = e^x \cos y + 2y$

v bodě $a = (0, 0)$ podle vektoru $v = (-1, 2)$.

$$\frac{\partial f}{\partial v}(a) = v \cdot \operatorname{grad} f(a) = (-1, 2) \cdot (1, 2) = -1 + 4 = \underline{\underline{3}}$$

$$\operatorname{grad} f(x, y) = \left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right)$$

$$= \left(e^x \cos y, -e^x \sin y + 2 \right)$$

$$\operatorname{grad} f(0, 0) = (1, 2)$$

$(\operatorname{arctg} x)' = \frac{1}{1+x^2}$

Určete všechny směry u, ve kterých je růst funkce $f(x, y) = \operatorname{arctg}(\frac{x}{y})$ v bodě $(1, 1)$ největší, nejmenší, nulový.

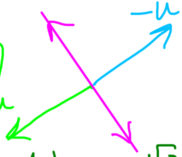
$u = \operatorname{grad} f(1,1) = \boxed{\left(\frac{1}{2}, -\frac{1}{2}\right)}$ $\quad -u = -\operatorname{grad} f(1,1) = \left(-\frac{1}{2}, \frac{1}{2}\right)$

$\frac{\partial f}{\partial x} = \frac{1}{1+(\frac{x}{y})^2}\cdot\frac{1}{y}$, $\quad \frac{\partial f}{\partial y} = \frac{1}{1+(\frac{x}{y})^2}\cdot\frac{-x}{y^2}$

nulový růst
$\left(\frac{1}{2}, \frac{1}{2}\right)$
$\left(-\frac{1}{2}, -\frac{1}{2}\right)$

Spočtěte velikost změny funkce ve směrech největšího a nejmenšího růstu.

$\frac{\partial f}{\partial h}(a) = h\cdot\operatorname{grad} f(a)$, $\|h\| = 1$



$\frac{u}{\|u\|} = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ $\quad \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{\sqrt{2}}{2}$

$\|u\| = \left\|\left(\frac{1}{2}, -\frac{1}{2}\right)\right\| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$ $\quad \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\cdot\left(\frac{1}{2}, -\frac{1}{2}\right) = -\frac{\sqrt{2}}{2}$

$L(h) = df(a)[h] = h \cdot \operatorname{grad} f(a)$

DÚ

lin. zobr.

Nalezněte první diferenciál $\boxed{\mathrm{d}\, f(a)}$, gradient $\operatorname{grad} f(a)$, rovnice tečné roviny a normály ke grafu funkce $f(x,y) = \sqrt{x - \sqrt{y}}$ v bodech $a = (2,1)$ a $b = (0,9)$. $\in D(f)$

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x-\sqrt{y}}}, \quad \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x-\sqrt{y}}} \cdot \frac{-1}{2\sqrt{y}}$$

$$\operatorname{grad} f(2,1) = \left(\frac{1}{2}, -\frac{1}{4}\right)$$ matice zobr. $df(a)$

$$df(a)[h] = (h_1\ h_2)\left(\frac{1}{2}, -\frac{1}{4}\right) = \frac{1}{2}h_1 - \frac{1}{4}h_2$$

$h = (h_1, h_2)$