Matematická analýza 2

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$n_1 = (2,2,0) \perp u_1$

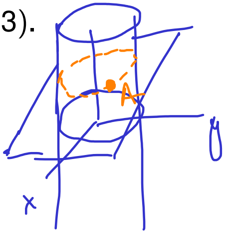
$n_2 = (1,0,1) \perp u_1$

$$u_1 = n_1 \times n_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (2, -2, -2) = 2e_1 - 2e_2 - 2e_3$$

Nalezněte tečnou přímku ke křivce zadané jako průnik dvou ploch

$S_1 \colon x^2 + y^2 = 2$ a $S_2 \colon x + z = 4$ v bodě $A = (1, 1, 3)$.

válec   rovina



$F(x,y,z) = x^2 + y^2 - 2$

$\operatorname{grad} F = (2x, 2y, 0)\big|_A = (2,2,0) = n_1$

$((x,y,z) - A) \cdot n_1 = 0$

$(x-1, y-1, z-3) \cdot (2,2,0) = 0$, $2x - 2 + 2y - 2 = 0$

$T \colon x + y - 2 = 0$ tečná rovina S_1

$T \cap S_2$! $\begin{cases} x + y = 2 \\ x + z = 4 \end{cases}$

$\begin{cases} x = t \\ y = 2 - t \\ z = 4 - t \end{cases}$

směrový vektor $(1,-1,-1) = u = \frac{1}{2} u_1$

Bod $B = (0,2,4)$

$B + u = (1,1,3) = A$

Najděte derivaci funkce $z = f(x, y)$, která splňuje rovnici

$$z^3 - 3xyz = 2.$$

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$?

$\nabla = \text{grad}$

Spočtěte $\nabla f(1, 1)$. $= \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\Big|_{(1,1,2)} = \left(\frac{2}{3}, \frac{2}{3}\right)$

$$\frac{\partial}{\partial x}\left(z^3 - 3xyz\right) = 3z^2 \frac{\partial z}{\partial x} - 3y\left(z + x \cdot \frac{\partial z}{\partial x}\right) = 0$$

$$z^2 \frac{\partial z}{\partial x} - yz - xy\frac{\partial z}{\partial x} = 0$$

$z = f(1,1) = 2$

$z^3 - 3z = 2$

$8 - 6 = 2$

$$\frac{\partial z}{\partial x} = \frac{yz}{z^2 - xy}, \quad z^2 - xy \neq 0$$

$$\frac{\partial z}{\partial y} = \frac{xz}{z^2 - xy}, \quad z^2 - xy \neq 0$$

$f(t), \quad t = x^2 - y^2$

Ať $F(x, y) = yf(x^2 - y^2)$, kde $f \in C^1$. Ukažte, že F vyhovuje rovnici

$F: \mathbb{R}^2 \to \mathbb{R}$

$$\frac{1}{x}\frac{\partial F}{\partial x} + \frac{1}{y}\frac{\partial F}{\partial y} = \frac{F}{y^2}$$

$f: \mathbb{R} \to \mathbb{R}$

$$\frac{\partial F}{\partial x} = \frac{\partial\left(y f(x^2 - y^2)\right)}{\partial x} = y \cdot f' \cdot 2x$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(y f(x^2 - y^2)\right) = f + y f' \cdot (-2y)$$

$$\frac{1}{x}\frac{\partial F}{\partial x} + \frac{1}{y}\frac{\partial F}{\partial y} = \frac{2xyf'}{x} + \frac{f - 2y^2 f'}{y} =$$

$$= 2\cancel{y}f' + \frac{f}{y} - 2\cancel{y}f' = \frac{f}{y} = \frac{yf}{y^2} = \frac{F}{y^2}$$

pozit. def. min | neg. def max | indef. | extr. není | kriterium nevhodné

$D_1>0,\ D_2>0,\ D_3>0$ | $D_1<0,\ D_2>0,\ D_3<0$ | $D_n \neq 0$

Nalezněte lokální extremy $f(x, y) = x^3 + y^3 - 3xy$. $D(f)=\mathbb{R}^2$

$\frac{\partial f}{\partial x} = 3x^2 - 3y = 0$ $\quad y = x^2$ $\quad y = y^4$ $\quad y^4 - y = 0$

$\frac{\partial f}{\partial y} = 3y^2 - 3x = 0$ $\quad x = y^2$ $\quad y(y^3-1) = 0$

$y_1 = 0,\ y_2 = 1$

$x_1 = 0,\ x_2 = 1$

stac. bod $A = (0,0)$, $B(1,1)$

$\frac{\partial^2 f}{\partial x^2} = 6x$, $\frac{\partial^2 f}{\partial x \partial y} = -3$, $\frac{\partial^2 f}{\partial y^2} = 6y$

$H = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$,

$H_A = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ $\quad D_1 = 0$, $D_2 = -9 \neq 0$ $\quad$ extr. není

$H_B = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ $\quad D_1 = 6 > 0$, $D_2 = 36 - 9 = 27 > 0$

pozit. def $\Rightarrow$ lok. min

$$d^2 f(0,0)[h,h] = (h_1, h_2)\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = -6 h_1 h_2$$

$h = (h_1, h_2)$

$(-1,1)$, $-6h_1h_2 > 0$

$(1,1)$, $-6h_1h_2 < 0$

$$d^2 f(1,1)[h,h] = (h_1, h_2)\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 6h_1^2 - 6h_1h_2 + 6h_2^2 > 0$$

$\text{pro } (h_1, h_2) \neq (0,0)$

$$= 6\left(h_1^2 - h_1h_2 + \frac{1}{4}h_2^2\right) - 6\cdot\frac{1}{4}h_2^2 + 6h_2^2$$

$$= 6\left(h_1 - \frac{1}{2}h_2\right)^2 + 6\cdot\frac{3}{4}h_2^2$$

1) $x^2 + y^2$

2) $-x^2 - y^2$

3) $x^2 - y^2$

$D(f) = \mathbb{R}^2$

Nalezněte lokální extremy $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$.

$\frac{\partial f}{\partial x} = 6x^2 + y^2 + 10x = 0$

$\frac{\partial f}{\partial y} = 2xy + 2y = 0$ $\quad y(x+1) = 0$ $\quad \boxed{y=0}$ nebo $\boxed{x=-1}$

1) $y = 0$, $\quad 6x^2 + 10x = 0$, $\quad x(6x+10) = 0$

$x_1 = 0$, $\quad x_2 = -\frac{5}{3}$

2) $x = -1$, $\quad 6 + y^2 - 10 = 0$, $\quad y^2 = 4$, $\quad y = \pm 2$

Stac. body $\quad A = (0,0)$, $B = (-\frac{5}{3}, 0)$, $C = (-1, 2)$, $D = (-1, -2)$

$\frac{\partial^2 f}{\partial x^2} = 12x + 10$, $\quad \frac{\partial^2 f}{\partial x \partial y} = 2y$, $\quad \frac{\partial^2 f}{\partial y^2} = 2x + 2$

$$H = \begin{pmatrix} 12x+10 & 2y \\ 2y & 2x+2 \end{pmatrix}$$

$H_A = \begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix}$ $D_1 = 10 > 0$, $D_2 = 20 > 0$ posit. def $\Rightarrow$ A je lok. min

$H_B = \begin{pmatrix} -10 & 0 \\ 0 & -\frac{4}{3} \end{pmatrix}$ $D_1 = -10 < 0$, $D_2 = \frac{40}{3} > 0$ negat. def $\Rightarrow$ B je lok. max

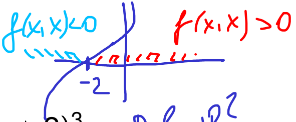
$H_C = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix}$ $D_1 = -2 < 0$, $D_2 = -16 < 0$ indef. $\Rightarrow$ v C není extrém

sedlové body

$H_D = \begin{pmatrix} -2 & -4 \\ -4 & 0 \end{pmatrix}$ $D_1 = -2 < 0$, $D_2 = -16 < 0$ indef $\Rightarrow$ v D není extrém

$f(-2,-2)=0$

na přímce $y=x$ máme $f(x,x)=(x+2)^3$



Nalezněte lokální extremy $f(x,y)=(y-x)^2+(y+2)^3$. $D_f=\mathbb{R}^2$

$$\frac{\partial f}{\partial x} = 2(y-x)\cdot(-1) = 0 \qquad x=y \qquad x=-2$$

$$\frac{\partial f}{\partial y} = 2(y-x)+3(y+2)^2=0 \qquad y+2=0,\ y=-2$$

stac. bod $A=(-2,-2)$

$$\frac{\partial^2 f}{\partial x^2}=2,\quad \frac{\partial^2 f}{\partial x\partial y}=-2,\quad \frac{\partial^2 f}{\partial y^2}=2+6(y+2)$$

$$H_A=\begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix} \qquad D_1=2,\ D_2=0$$

kriterium nerozhodne

lok. extrem není

$$d^2f(-2,-2)[h,h]=2h_1^2-4h_1h_2+2h_2^2=2(h_1-h_2)^2\geq 0$$

$=0$ pro $h_1=h_2$