Matematická analýza 2

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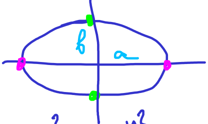


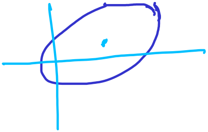
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

střed je v $(0,0)$

Najděte poloosy elipsy $\boxed{7x^2 - 6xy + 7y^2 = 8.}$

$x = 0 \quad 7y^2 = 8, \quad y = \pm\sqrt{\frac{8}{7}}$

$y = 0 \quad 7x^2 = 8 \quad x = \pm\sqrt{\frac{8}{7}}$



$f(x,y) = x^2 + y^2$

$g(x,y) = 7x^2 - 6xy + 7y^2 - 8$

$L = f - \lambda g = x^2 + y^2 - \lambda(7x^2 - 6xy + 7y^2 - 8)$

$$\boxed{\frac{\partial L}{\partial \lambda} = -(7x^2 - 6xy + 7y^2 - 8) = 0}$$

vzdálenost (x,y) od $(x_0,y_0) = (0,0)$

$d = \|(x,y) - (x_0,y_0)\| = \sqrt{(x-x_0)^2 + (y-y_0)^2} = \boxed{\sqrt{x^2+y^2}}$

min

max

$$\frac{\partial L}{\partial x} = 2x - \lambda(14x - 6y) = 2x - 14\lambda x + 6\lambda y = 0 \quad /:2$$

$$\frac{\partial L}{\partial y} = 2y - \lambda(-6x + 14y) = 6\lambda x + 2y - 14\lambda y = 0 \quad /:2$$

$$7x^2 - 6xy + 7y^2 = 8$$

$$x(1 - 7\lambda) = -3\lambda y \quad (1)$$

$$y(1 - 7\lambda) = -3\lambda x \quad (2)$$

$x = 0$ nebo $\lambda = 0$

$x = y = 0$

z (1) $\boxed{x = -\frac{3\lambda y}{1 - 7\lambda}}$

$$y(1 - 7\lambda) = -3\lambda \frac{-3\lambda}{1 - 7\lambda} y$$

$$y(1 - 7\lambda)^2 = 9\lambda^2 y$$

$$y(1 - 14\lambda + 49\lambda^2 - 9\lambda^2) = 0$$

1) $y = 0$, $x = 0$ není na elipse

2) $40\lambda^2 - 14\lambda + 1 = 0$

$$D = 14^2 - 4 \cdot 40 = 4 \cdot 7^2 - 4 \cdot 40 = 4 \cdot 9$$

$$\lambda_{1,2} = \frac{14 \pm 6}{80} < \begin{matrix} \lambda_1 = \frac{1}{4} \\ \lambda_2 = \frac{1}{10} \end{matrix}$$

1) $\lambda = \frac{1}{4}$, $x = -\frac{3/4}{-3/4} y$

$\boxed{x = y}$

2) $\lambda_2 = 0{,}1$, $x = -\frac{0{,}3}{0{,}3} y$ $\boxed{x = -y}$

1) $x = y$, $7x^2 - 6x^2 + 7x^2 = 8$, $8x^2 = 8$, $x^2 = 1$

$x = \pm 1$

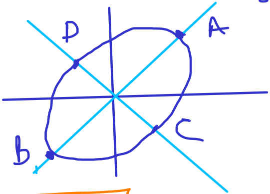
stac. body $A = (1,1)$, $B(-1,-1)$

2) $x = -y$, $7x^2 + 6x^2 + 7x^2 = 8$, $20x^2 = 8$, $x^2 = \frac{2}{5}$

$x = \pm\sqrt{\frac{2}{5}}$

stac. body $C = \left(\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}\right)$

$D = \left(-\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}\right)$



$d = \sqrt{x^2 + y^2}$

$d(A) = \sqrt{2}$

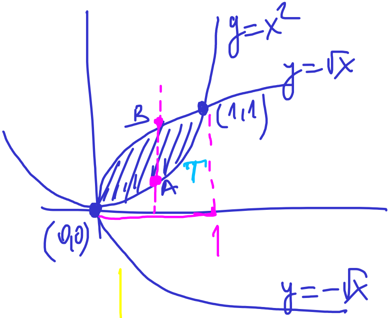
$d(B) = \sqrt{2}$

$d(C) = d(D) = \sqrt{\frac{2}{5} + \frac{2}{5}} = \frac{2}{\sqrt{5}}$

poloosy jsou $\frac{2}{\sqrt{5}}$ a $\sqrt{2}$

$y = x^2 \qquad x = y^2$

$$\iint_T xy, \text{ kde } T = \{(x,y) \mid x^2 \le y,\ y^2 \le x\}.$$



$$\iint_T xy = \int_0^1 \underbrace{\left(\int_{x^2}^{\sqrt{x}} xy \, dy \right)}\, \underline{dx} =$$

v A: $y = x^2$

v B: $y = \sqrt{x}$

$$= \int_0^1 x \left[\frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx = \frac{1}{2} \int_0^1 x(x - x^4)\, dx$$

$$= \frac{1}{2} \int_0^1 (x^2 - x^5)\, dx = \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^6}{6} \right]_0^1 = \frac{1}{12}$$

$$\frac{1}{2}\left(\frac{1}{3} - \frac{1}{6}\right)$$

$y = x^2, \quad x = y^2$

$y = (y^2)^2$

$y^4 - y = 0$

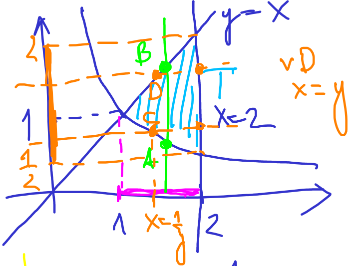
$y(y^3 - 1) = 0$

$y = 0, \quad y = 1$

$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{y}}^{2}\left(\frac{x}{y}\right)^2 dx\,dy + \int_{1}^{2}\int_{y}^{2}\left(\frac{x}{y}\right)^2 dx\,dy$$

$$y = \frac{1}{x}$$

$$\iint_T \left(\frac{x}{y}\right)^2, \text{ kde } \partial T\ :\ x = 2,\ y = x,\ xy = 1.$$



$$\int_1^2 \int_{\frac{1}{x}}^{x} \left(\frac{x}{y}\right)^2 dy\, dx = \int_1^2 x^2 \left[-\frac{1}{y}\right]_{\frac{1}{x}}^{x} dx$$

$$\int \frac{1}{y^2} dy = \int y^{-2} dy = \frac{y^{-1}}{-1} + c$$

$y = x \quad y = \frac{1}{x}$

$x = \frac{1}{x}, \quad x^2 = 1$

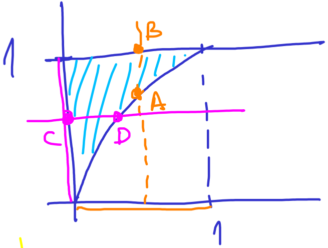
$x = \pm 1$

$$= \int_1^2 x^2 \left(-\frac{1}{x} + x\right) dx = \int_1^2 \left(-x + x^3\right) dx = \left[-\frac{x^2}{2} + \frac{x^4}{4}\right]_1^2 =$$

$$= -2 + 4 - \left(-\frac{1}{2} + \frac{1}{4}\right) = 2 + \frac{1}{4} = \frac{9}{4}$$

Oy

$$\iint_T e^{\frac{x}{y}}, \text{ kde } \partial T : x=0,\ y=1,\ x=y^2.$$



v C $x=0$

v D $x=y^2$

$$\int_0^1\int_{\sqrt{x}}^1 e^{\frac{x}{y}}\,dy\,dx \qquad \int e^{\frac{1}{y}}\,dy$$

$$\int_0^1\int_0^{y^2} e^{\frac{x}{y}}\,dx\,dy = \int_0^1 \left[y e^{\frac{x}{y}}\right]_0^{y^2} dy = \int_0^1 (ye^y - y)\,dy =$$

$$= \left[ye^y - e^y - \frac{y^2}{2}\right]_0^1 = \left(\cancel{e} - \cancel{e} - \frac{1}{2}\right) - (0-1-0) = \frac{1}{2}$$

$\int uv'$

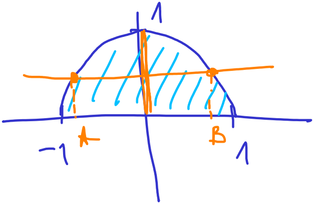
$$\int e^{\frac{x}{5}}\,dx = 5e^{\frac{x}{5}} + C, \quad \int ye^y\,dy = \left| \begin{matrix} u=y & v'=e^y \\ u'=1 & v=e^y \end{matrix} \right| =$$

$uv \quad - \quad \int u'v$

$$= ye^y - \int e^y dy = ye^y - e^y + C$$

Změňte pořadí integrace v integrálu

$$\int_{-1}^{1}\left(\int_{0}^{\sqrt{1-x^2}} f\,dy\right)dx = \int_{0}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f\,dx\,dy$$



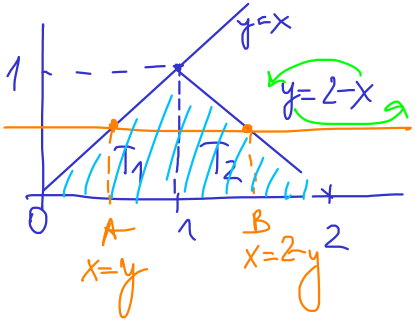
$y=0$

$y=\sqrt{1-x^2}$

$y^2=1-x^2, \quad x^2+y^2=1, \quad x^2=1-y^2, \quad x=\pm\sqrt{1-y^2}$

Změňte pořadí integrace v integrálu

$$\underbrace{\int_0^1 \left(\int_0^x f \, dy \right) dx}_{T_1} + \underbrace{\int_1^2 \left(\int_0^{2-x} f \, dy \right) dx}_{T_2}$$



$$\int_0^1 \int_y^{2-y} f \, dx \, dy$$

$m = \iint_T \rho \quad (\rho = x) \qquad x_t = \dfrac{\iint_T x\rho}{\iint_T \rho} \quad (x\rho = x^2) \qquad y_t = \dfrac{\iint_T y\rho}{\iint_T \rho} \quad (y\rho = xy)$

Vypočtěte hmotnost trojúhelníku se vrcholy $(0,0)$, $(1,1)$, $(4,0)$, jehož plošná hustota je $\rho(x,y) = x$. Zjistěte jeho těžiště. (x_t, y_t)

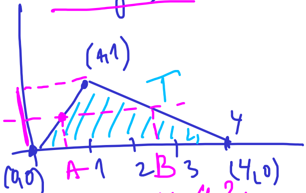
$$m = \iint_T x = \int_0^1 \int_y^{4-3y} x \, dx \, dy = \int_0^1 \left[\frac{x^2}{2}\right]_y^{4-3y} dy =$$

$$= \frac{1}{2}\int_0^1 \left((4-3y)^2 - y^2\right) dy$$

$$= \frac{1}{2}\left[\frac{(4-3y)^3}{3(-3)} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{-9} - \frac{1}{3} - \frac{4^3}{-9}\right) = \frac{1}{2}\cdot\frac{-1-3+64}{9} = \frac{10}{3}$$

$$\boxed{m = \frac{10}{3}}$$



$x = y \qquad x = 4-3y$

směr. vektor

$(4,0)-(1,1) = (3,-1)$

$x = 1 + t\cdot 3$

$y = 1 + t\cdot(-1)$

$t = \dfrac{x-1}{3} = \dfrac{y-1}{-1}$

$x = 4-3y$

$$\iint_T x\rho = \int_0^1\int_y^{4-3y} x^2\,dx\,dy = \int_0^1 \left[\frac{x^3}{3}\right]_y^{4-3y} dy =$$

$$X_t = \frac{21}{10}$$

$$= \frac{1}{3}\int_0^1 \left((4-3y)^3 - y^3\right) dy = \frac{1}{3}\left[\frac{(4-3y)^4}{4\cdot(-3)} - \frac{y^4}{4}\right]_0^1 =$$

$$= \frac{1}{3}\left(\frac{1}{-12} - \frac{1}{4} - \frac{4^4}{-12}\right) = \frac{1}{3}\,\frac{-1-3+64\cdot 4}{12} = \frac{1}{3}\cdot\frac{63}{3} = \frac{21}{3}$$

$$\iint_T y\rho = \int_0^1\int_y^{4-3y} xy\,dx\,dy = \int_0^1 y\left[\frac{x^2}{2}\right]_y^{4-3y} dy =$$

$$= \frac{1}{2}\int_0^1 y\left((4-3y)^2 - y^2\right) dy = \frac{1}{2}\int_0^1 y\left(16-24y+9y^2-y^2\right) dy =$$

$$= \int_0^1 \left(8y - 12y^2 + 4y^3\right) dy = \left[8\frac{y^2}{2} - 12\frac{y^3}{3} + 4\frac{y^4}{4}\right]_0^1 = 4-4+1 = 1$$

$$y_t = \frac{3}{10}$$