Matematická analýza 2

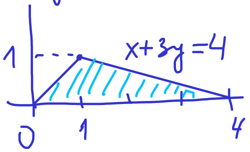
Natalie Žukovec

8. cvičení 2021

$$m = \iint_T \rho\,, \quad x_t = \frac{1}{m}\iint_T x\rho\,, \quad y_t = \frac{1}{m}\iint_T y\rho$$

Vypočtěte hmotnost trojúhelníku se vrcholy $(0,0)$, $(1,1)$, $(4,0)$, jehož plošná hustota je $\rho(x,y) = x$. Zjistěte jeho těžiště. (x_t, y_t)

$$\int_0^1\int_y^{4-3y} x \,dx\,dy = \int_0^1 \left[\frac{x^2}{2}\right]_y^{4-3y} dy =$$



$$= \frac{1}{2}\int_0^1 \left((4-3y)^2 - y^2\right) dy = \frac{1}{2}\left[\frac{(4-3y)^3}{3\,(-3)} - \frac{y^3}{3}\right]_0^1 =$$

$$= \frac{1}{2}\left(\frac{1}{-9} - \frac{1}{3} - \frac{4^3}{-9}\right) = \frac{1}{2}\cdot\frac{-1-3+64}{9} = \frac{10}{3},$$

$$\boxed{m = \frac{10}{3}}$$

$$\iint_T x\rho = \int_0^1\int_y^{4-3y} x^2\,dx\,dy = \int_0^1 \left[\frac{x^3}{3}\right]_y^{4-3y} dy =$$

$$x_t = \frac{21}{10}$$

$$= \frac{1}{3}\int_0^1 \left((4-3y)^3 - y^3\right) dy = \frac{1}{3}\left[\frac{(4-3y)^4}{4\cdot(-3)} - \frac{y^4}{4}\right]_0^1 =$$

$$= \frac{1}{3}\left(\frac{1}{-12} - \frac{1}{4} - \frac{4^4}{-12}\right) = \frac{1}{3}\,\frac{-1-3+64\cdot 4}{12} = \frac{1}{3}\cdot\frac{63}{3} = \frac{21}{3}$$

$$\iint_T y\rho = \int_0^1\int_y^{4-3y} xy\,dx\,dy = \int_0^1 y\left[\frac{x^2}{2}\right]_y^{4-3y} dy =$$

$$= \frac{1}{2}\int_0^1 y\left((4-3y)^2 - y^2\right) dy = \frac{1}{2}\int_0^1 y\left(16-24y+9y^2-y^2\right) dy =$$

$$= \int_0^1 \left(8y - 12y^2 + 4y^3\right) dy = \left[8\frac{y^2}{2} - 12\frac{y^3}{3} + 4\frac{y^4}{4}\right]_0^1 = 4-4+1 = 1$$

$$y_t = \frac{3}{10}$$

$2x-6=0$

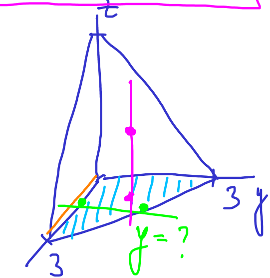
Vypočtěte $\iiint\limits_P y$, kde ∂P : $\underbrace{x=0,\ y=0,\ z=0}_{\text{roviny}}$, $2x+2y+z-6=0$.

$$\int_0^3\int_0^{3-x}\int_0^{6-2x-2y} y\,dz\,dy\,dx =$$

$$= \int_0^3\int_0^{3-x} y(6-2x-2y)\,dy\,dx$$

$$= \int_0^3\int_0^{3-x} \left((6-2x)y-2y^2\right) dy\,dx =$$

$$= \int_0^3 \left[(6-2x)\frac{y^2}{2}-2\frac{y^3}{3}\right]_0^{3-x} dx =$$



pro $z=0$

$2x+2y-6=0$

$y=3-x$

$$= \int_0^3 \left[(3-x)^3 - \frac{2}{3}(3-x)^3 \right] dx = \frac{1}{3}\int_0^3 (3-x)^3 dx =$$

$$= \frac{1}{3}\left[-\frac{(3-x)^4}{4} \right]_0^3 = \frac{3^4}{3 \cdot 4} = \frac{27}{4}$$

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \left| \begin{matrix} x = \varphi(t) \\ dx = \varphi'(t)\,dt \end{matrix} \right| = \int_\alpha^\beta \underbrace{f(\varphi(t))}\,\underbrace{\varphi'(t)}\,dt$$

Za použití substituce $u = x + 2y$, $v = x - y$ vypočtěte integrál

$$\int_0^{2/3} \int_y^{2-2y} (x+2y)e^{y-x}\,dx\,dy = \int_0^2 \int_0^u \underbrace{u e^{-v}}\,\underbrace{\tfrac{1}{3}}\,dv\,du$$

(1) $u = x + 2y$

(2) $v = x - y$

$$\boxed{\begin{aligned} x &= \tfrac{1}{3}u + \tfrac{2}{3}v \\ y &= \tfrac{1}{3}u - \tfrac{1}{3}v \end{aligned}}$$

(1)−(2): $u - v = 3y$

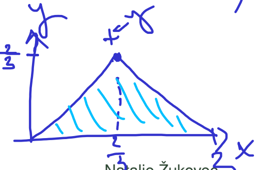
(2): $x = v + y = v + \frac{1}{3}u - \frac{1}{3}v$

$\Phi(u,v) = (x,y)$

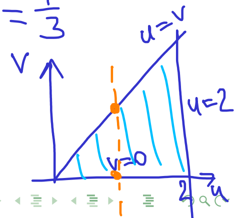
$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$J_\Phi = \begin{pmatrix} \operatorname{grad} x \\ \operatorname{grad} y \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$J_\Phi = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}, \quad \Delta_\Phi = |\det J_\Phi| = \left| -\frac{1}{9} - \frac{2}{9} \right| = \frac{1}{3}$$



$$\left[\begin{array}{lll} x = y, & x - y = 0, & v = 0 \\ x = 2 - 2y, & x + 2y = 2, & u = 2 \\ y = 0 & & u - v = 0 \end{array} \right]$$



$$\int_0^2 \int_0^u u e^{-v} \cdot \frac{1}{3} \, dv \, du = \int_0^2 \frac{1}{3} \cdot u \left[-e^{-v}\right]_0^u du =$$

$$= \int_0^2 \frac{1}{3} u \left[-e^{-u} - (-1)\right] du = \frac{1}{3} \int_0^2 \left(u - u e^{-u}\right) du =$$

$$= \frac{1}{3} \left[\frac{u^2}{2} + u e^{-u} + e^{-u}\right]_0^2 = \frac{1}{3} \left(2 + 2e^{-2} + e^{-2} - 0 - 0 - 1\right)$$

$$= \frac{1}{3} + e^{-2}$$

$$\int e^{-v} dv = -e^{-v} + C$$

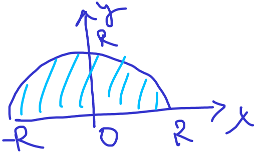
$$\int u e^{-u} du = \left| \begin{matrix} f = u & g' = e^{-u} \\ f' = 1 & g = -e^{-u} \end{matrix} \right| = -u e^{-u} + \int + e^{-u} du$$

$$= -u e^{-u} - e^{-u} + C$$

$$\int f g' = f g - \int f' g$$

Pomocí polárních souřadnic vypočtěte $\iint_T \sqrt{R^2 - x^2 - y^2}$, kde

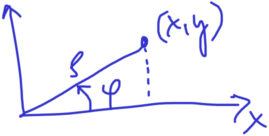
$T = \{(x, y) \mid x^2 + y^2 \leq R^2,\ y \geq 0\}$.

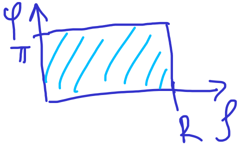


$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \end{cases}$$

$\rho \geq 0$

$\varphi \in \langle 0, 2\pi)$





$\Delta = \rho$

$f(x,y) = \sqrt{R^2 - x^2 - y^2} = \sqrt{R^2 - \rho^2}$

$x^2 + y^2 = \rho^2 \cos^2\varphi + \rho^2 \sin^2\varphi = \rho^2$

$$\int_0^{\pi}\int_0^{R} \sqrt{R^2-\rho^2} \cdot \rho \, d\rho \, d\varphi = \int_0^R \sqrt{R^2-\rho^2} \cdot \rho \, d\rho \cdot \overbrace{\int_0^{\pi} 1 \, d\varphi}^{= \pi} =$$

$$= \left| \begin{array}{l} t = R^2 - \rho^2 \\ dt = -2\rho\, d\rho \\ \rho\, d\rho = -\frac{1}{2}\, dt \end{array} \right| = -\frac{\pi}{2} \int_{R^2}^{0} \sqrt{t}\, dt = \frac{\pi}{2} \int_{0}^{R^2} \sqrt{t}\, dt =$$

$$\rho = 0 \;\ldots\; t = R^2$$

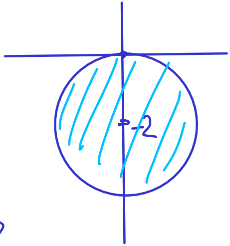
$$\rho = R \;\ldots\; t = 0$$

$$= \frac{\pi}{2} \left[\frac{t^{\frac{3}{2}}}{3/2} \right]_0^{R^2} = \frac{\pi}{3} \cdot R^3$$

$$\int_0^{2\pi} \sin\varphi \, d\varphi = 0$$

Pomocí polárních souřadnic vypočtěte $\iint_T x^2 + y^2$, kde

$T = \{(x, y) \mid x^2 + (y + 2)^2 \le 4\}$.



$x = \rho \cos\varphi$

$y + 2 = \rho \sin\varphi$

$y = \rho \sin\varphi - 2$

$\Delta = \rho$

$\rho \in \langle 0, 2\rangle$

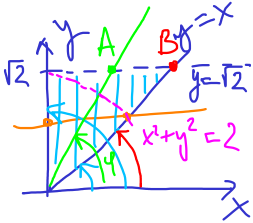
$\varphi \in \langle 0, 2\pi\rangle$

$f(x,y) = x^2 + y^2 = \rho^2 \cos^2\varphi + (\rho \sin\varphi - 2)^2 = \rho^2 - 4\rho\sin\varphi + 4$

$$\int_0^2 \int_0^{2\pi} (\rho^2 - \underbrace{4\rho \sin\varphi}_{0} + 4)\rho \, d\varphi \, d\rho = \int_0^2 (\rho^3 + 4\rho) \cdot [\varphi]_0^{2\pi} \, d\rho$$

$$= 2\pi \left[\frac{\rho^4}{4} + 2\rho^2\right]_0^2 = 2\pi(4 + 8) = 24\pi$$

Proveďte přechod do polárních souřadnic a výsledné integrály napište v obou pořadích $\int_0^{\sqrt{2}} \int_0^{y} f \, dx \, dy = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\sqrt{2}}{\sin\varphi}} f \cdot \rho \, d\rho \, d\varphi$



$x = \rho \cos\varphi$
$y = \rho \sin\varphi$

$y = \sqrt{2} \Rightarrow \boxed{\rho \sin\varphi = \sqrt{2}}$

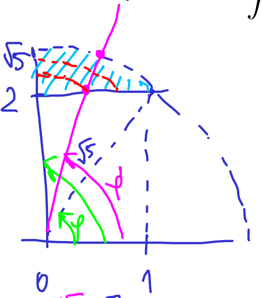
$\rho = \frac{\sqrt{2}}{\sin\varphi}$

$\sin\varphi = \frac{\sqrt{2}}{\rho}$

$\varphi = \arcsin\frac{\sqrt{2}}{\rho}$

$= \int_0^{\sqrt{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f \cdot \rho \, d\varphi \, d\rho + \int_{\sqrt{2}}^{2} \int_{\frac{\pi}{4}}^{\arcsin\frac{\sqrt{2}}{\rho}} f \cdot \rho \, d\varphi \, d\rho$

Proveďte přechod do polárních souřadnic a výsledné integrály napište v obou pořadích $\int_0^1 \int_2^{\sqrt{5-x^2}} f \, dy \, dx = \int_{\operatorname{arctg} 2}^{\frac{\pi}{2}} \int_{\frac{2}{\sin\varphi}}^{\sqrt{5}} f\rho \, d\rho \, d\varphi$



$y = \sqrt{5-x^2}$

$x^2 + y^2 = 5, \quad R = \sqrt{5}$

$\operatorname{tg}\varphi = 2$

$y = 2 = \rho \sin\varphi$

$\rho = \frac{2}{\sin\varphi}$

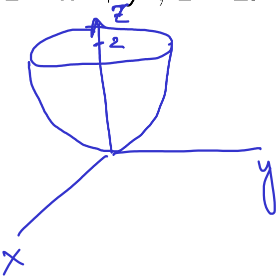
$\sin\varphi = \frac{2}{\rho}$

$\varphi = \arcsin\frac{2}{\rho}$

$= \int_2^{\sqrt{5}} \int_{\arcsin\frac{2}{\rho}}^{\frac{\pi}{2}} f\rho \, d\varphi \, d\rho$

Vypočtěte $\iiint\limits_P z^2$, kde P je omezena plochami $z = x^2 + y^2,\ z = 2.$

DÚ



Vypočtěte $\iiint\limits_P x^2 + y^2$, kde

$$P = \{(x, y, z) \mid R^2 \leq x^2 + y^2 + z^2 \leq 1,\ z \geq 0\}.$$