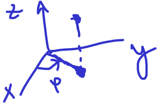
Matematická analýza 2

Natalie Žukovec

9. cvičení 2021

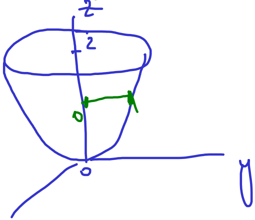


$z=\rho^2$, $\rho=\sqrt{z}$

Vypočtěte $\iiint\limits_P z^2$, kde P je omezena plochami $z = x^2+y^2$, $z = 2$.

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

$$\Delta = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z}=0 \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z}=0 \\ \frac{\partial z}{\partial \rho}=0 & \frac{\partial z}{\partial \varphi}=0 & \frac{\partial z}{\partial z}=1 \end{vmatrix} = \rho$$



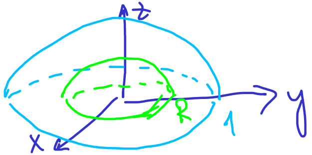
$$\int_0^2\int_0^{2\pi}\int_0^{\sqrt{z}} z^2\rho\, d\rho\, d\varphi\, dz = \int_0^2\int_0^{2\pi} z^2\left[\frac{\rho^2}{2}\right]_0^{\sqrt{z}} d\varphi\, dz =$$

$$= \int_0^2\int_0^{2\pi} z^2\left(\frac{z}{2}-0\right) d\varphi\, dz = \frac{1}{2}\int_0^2 z^3\cdot 2\pi\, dz = \pi\left[\frac{z^4}{4}\right]_0^2$$

$$\frac{z^3}{2}\underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

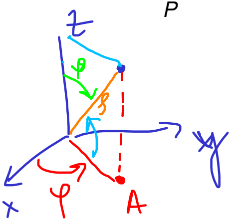
$$= 4\pi$$

Vypočtěte $\iiint\limits_P \underbrace{x^2 + y^2}_{f}$, kde



$$P = \{(x, y, z) \mid \underbrace{R^2 \le x^2 + y^2 + z^2 \le 1}, \ \underbrace{z \ge 0}\}.$$

$R \le 1$



$x = \rho \sin\psi \cos\varphi$

$y = \rho \sin\psi \sin\varphi$

$z = \rho \cos\psi$

$\Delta = \rho^2 \sin\psi$

$\rho \ge 0$

$\varphi \in \langle 0, 2\pi \rangle$

$\psi \in \langle 0, \pi \rangle$

obecně

pro P: $\rho \in \langle R, 1 \rangle$, $\varphi \in \langle 0, 2\pi \rangle$, $\psi \in \langle 0, \frac{\pi}{2} \rangle$

$f(x, y, z) = x^2 + y^2 = \rho^2 \sin^2\psi \underbrace{(\cos^2\varphi + \sin^2\varphi)}_{=1}$

$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_R^1 \rho^2 \sin^2\psi \cdot \rho^2 \sin\psi \, d\rho \, d\varphi \, d\psi =$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \sin^3\psi \left[\frac{\rho^5}{5}\right]_R^1 d\varphi\, d\psi = \frac{1}{5}(1-R^5)\int_0^{\frac{\pi}{2}}\underbrace{\int_0^{2\pi}\sin^3\psi\, d\varphi}\, d\psi$$

$$\sin^3\psi\int_0^{2\pi} d\varphi$$

$$= \frac{1}{5}(1-R^5)\int_0^{\frac{\pi}{2}}\sin^3\psi\cdot 2\pi\, d\psi = \Big/\ t=\cos\psi,\quad dt = -\sin\psi\, d\psi\ \Big/$$

$$= \frac{2\pi}{5}(1-R^5)\int_{0}^{1}(1-t^2)(+dt)$$

$$\psi=0,\quad t=1$$
$$\psi=\frac{\pi}{2},\quad t=0$$

$$\sin^3\psi = \sin^2\psi\cdot\sin\psi = (1-\cos^2\psi)\sin\psi$$

$$= \frac{2\pi}{5}(1-R^5)\underbrace{\left[t-\frac{t^3}{3}\right]_0^1}_{=\frac{2}{3}} = \frac{4\pi}{15}(1-R^5)$$

$I = \iiint_K v^2 \cdot \rho$, kde $v(x,y,z)$ je vzdálenost (x,y,z) od osy z

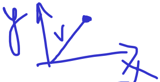
Vypočtěte moment setrvačnosti koule s poloměrem R vzhledem k tečné přímce (hustota $\rho = 1$).

$x = \rho \sin\psi \cos\varphi + R$

$y = \rho \sin\psi \sin\varphi$

$z = \rho \cos\psi$

$\Delta = \rho^2 \sin\psi$



$v = \sqrt{x^2+y^2}$

$v^2 = x^2 + y^2$

$I = \iiint_K x^2+y^2$

$\rho^2\sin^2\psi\cos^2\varphi + 2\rho\sin\psi\cos\varphi \cdot R + R^2$

$= \int_0^R \int_0^\pi \int_0^{2\pi} \left((\rho \sin\psi \cos\varphi + R)^2 + \rho^2 \sin^2\psi \sin^2\varphi \right) \rho^2 \sin\psi \, d\varphi \, d\psi \, d\rho$

$= \int_0^R \int_0^\pi \int_0^{2\pi} \left(\rho^2 \sin^2\psi + 2\rho R \sin\psi \cos\varphi + R^2\right) \rho^2 \sin\psi \, d\varphi \, d\psi \, d\rho$

$\int_0^{2\pi} \cos\varphi \, d\varphi = 0$

$\int_0^{2\pi} d\varphi$

$$= \int_0^R \int_0^\pi \left(\rho^4 \sin^3\psi + R^2\rho^2 \sin\psi\right)\cdot 2\pi \, d\psi \, d\rho = \ldots$$ DÚ

$$= 2\pi \int_0^R \int_1^{-1} \rho^4 (1-u^2)(-du)\, d\rho + 2\pi R^2 \int_0^R \rho^2 \left[-\cos\psi\right]_0^\pi d\rho =$$

$u = \cos\psi \qquad \psi = 0, \ u = 1$

$du = -\sin\psi \, d\psi \qquad \psi = \pi, \ u = -1$

$$= 2\pi \int_0^R \rho^4 \underbrace{\left[u - \frac{u^3}{3}\right]_{-1}^{1}} d\rho + 2\pi R^2 \int_0^R \rho^2 \left(-(-1)-(-1)\right) d\rho =$$

$$= 2\pi \frac{4}{3}\left[\frac{\rho^5}{5}\right]_0^R + 4\pi R^2 \left[\frac{\rho^3}{3}\right]_0^R = \frac{8\pi}{15}R^5 + \frac{4\pi}{3}R^5$$

$$1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right) = 2 - \frac{2}{3} = \frac{4}{3}$$

$$= \frac{28}{15}\pi R^5$$

$$\int_C f\,ds = \int_a^b f(\varphi(t))\,\|\varphi'(t)\|\,dt \qquad \text{parametrizace } \varphi: \langle a,b\rangle \to C$$

Určete $\underbrace{\text{délku spirály}}_{C}$ mající v polárních souřadnicích tvar $\rho = e^{-3\varphi}$, $\varphi \in \langle 0, 2\pi \rangle$.

$$\int_C 1\,ds$$

parametrizace!

$$x = \rho\cos\varphi = e^{-3\varphi}\cos\varphi$$
$$y = \rho\sin\varphi = e^{-3\varphi}\sin\varphi \qquad \varphi \in \langle 0, 2\pi\rangle$$

$$\Phi(\varphi) = (\overbrace{e^{-3\varphi}\cos\varphi}^{x}, \overbrace{e^{-3\varphi}\sin\varphi}^{y}), \quad \varphi \in \langle 0, 2\pi\rangle$$

$$\Phi'(\varphi) = (-3e^{-3\varphi}\cos\varphi + e^{-3\varphi}(-\sin\varphi), -3e^{-3\varphi}\sin\varphi + e^{-3\varphi}\cos\varphi)$$

$$\|\Phi'(\varphi)\| = e^{-3\varphi}\sqrt{(3\cos\varphi + \sin\varphi)^2 + (-3\sin\varphi + \cos\varphi)^2}$$

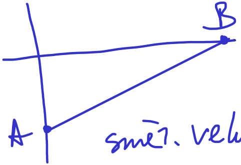
$$= e^{-3\varphi}\sqrt{9 + 1 + \cancel{2\cdot 3\cos\varphi\sin\varphi} + \cancel{2(-3\sin\varphi)\cos\varphi}}$$

$$= e^{-3\varphi}\sqrt{10}$$

$$\int_C 1\,ds = \int_0^{2\pi} e^{-3\varphi}\cdot\sqrt{10}\,d\varphi = \sqrt{10}\left[-\frac{1}{3}e^{-3\varphi}\right]_0^{2\pi} =$$

$$= \frac{\sqrt{10}}{3}\left(1 - e^{-6\pi}\right)$$

Vypočtěte křivkový integrál $\int_C \frac{1}{x-y}\,ds$, kde C je úsečka s krajními body $A = (0,-2)$, $B = (4,0)$.



směr. vekt. $B-A$

parametrizace C:

$$\varphi(t) = A + t(B-A), \quad t \in \langle 0, 1 \rangle$$

$$\varphi(0) = A$$

$$\varphi(1) = A + B - A = B$$

$$\varphi(t) = (0,-2) + t(4,2) = (\overbrace{4t}^{x}, \overbrace{-2+2t}^{y}), \quad t \in \langle 0, 1 \rangle$$

$$\varphi'(t) = (4,2), \quad \|\varphi'(t)\| = \sqrt{4^2+2^2} = \sqrt{20} = 2\sqrt{5}$$

$$\int_C \frac{1}{x-y}\,ds = \int_0^1 \frac{1}{4t-(-2+2t)} \cdot 2\sqrt{5}\,dt = 2\sqrt{5}\int_0^1 \frac{1}{2t+2}\,dt$$

$$= \sqrt{5}\int_0^1 \frac{dt}{t+1} = \sqrt{5}\Big[\ln|t+1|\Big]_0^1 = \sqrt{5}(\ln 2 - \ln 1)$$

$$= \sqrt{5}\ln 2$$

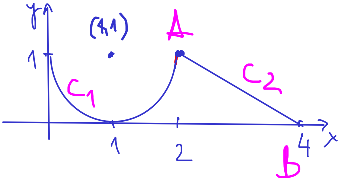
$$\int_{C_1 \cup C_2} \tau \, ds = \int_{C_1} \tau \, ds + \int_{C_2} \tau \, ds$$

Vypočtěte hmotnost křivky C nakreslené na obrázku, jestliže jeji délková hustota v bodě $(x, y) \in C$ je dána funkci $\tau(x, y) = x^2 + y^2$.

C_1 parametrizace:

$$\boxed{\begin{aligned} x &= \cos t \\ y &= \sin t \\ t &\in \langle 0, \pi \rangle \end{aligned}} \quad \begin{aligned} &+1 \\ &+1 \end{aligned}$$

$t \in \langle \pi, 2\pi \rangle$



C_2: $A = (2,1)$, $B = (4,0)$

DÚ