Teorie algoritmů

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$$\lim_{n\to\infty} \frac{5^n+n^4}{5^n} = \lim_{n\to\infty}\frac{5^n}{5^n} + \lim_{n\to\infty}\frac{n^4}{5^n} = 1+0=1$$

Z definice ukažte, že platí: Pro $f(n) = 5^n + n^4$ máme $f(n) \in \Theta(5^n)$.

$\exists c_1, c_2 > 0 \ \exists n_1 \in \mathbb{N} \ \forall n \geq n_1 \quad c_1 5^n \leq 5^n + n^4 \leq c_2 5^n$

$\in \mathbb{R}$

$c_1 = 1$

$c_2 = 2$

$n_1 = 1$

indukce:

pro $n=1$, $1^4 \leq 5^1$ platí

pokud $n^4 \leq 5^n$, pak

$(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 \leq 5n^4 \stackrel{i.p.}{\leq} 5\cdot 5^n = 5^{n+1}$

$c_1 = 1$, protože $n^4 \geq 0$

$c_2 = 2$

$5^n + n^4 \leq 2\cdot 5^n$

$\boxed{n^4 \leq 5^n \text{ pro } n \geq 1}$

Jsou dány tři nezáporné funkce $f(n)$, $g(n)$ a $h(n)$. Dokažte:

Jestliže $f(n)$ je $\mathcal{O}(g(n))$ a $g(n)$ je $\mathcal{O}(h(n))$, tak $f(n)$ je $\mathcal{O}(h(n))$.

$$\exists c_1 > 0 \ \exists n_1 \in \mathbb{N} \ \forall n \geq n_1 \quad f(n) \leq c_1 g(n)$$

$$\exists c_2 > 0 \ \exists n_2 \in \mathbb{N} \ \forall n \geq n_2 \quad c_1 g(n) \leq c_1 c_2 h(n)$$

chceme

$$\exists c_3 > 0 \ \exists n_3 \in \mathbb{N} \ \forall n \geq n_2 \quad f(n) \leq c_3 h(n)$$

$$f(n) \leq c_1 g(n) \leq c_1 c_2 h(n)$$

$c_3 = c_1 c_2$

$n_3 = \max\{n_1, n_2\}$

$(f_1+f_2)(n) = f_1(n)+f_2(n)$

Dokažte následující tvrzení:

Jsou dány nezáporné funkce $f_1(n)$, $f_2(n)$ *a* $h(n)$ *takové, že*

$f_1(n) \in \Theta(h(n))$, $f_2(n) \in \Theta(h(n) \lg n)$, *pak* $(f_1+f_2)(n) \in \Theta(h(n) \lg n)$.

$\exists c_1\, c_2 \quad n_1$ $\qquad$ $c_1 h(n) \leq f_1(n) \leq c_2 h(n)$

$\exists d_1\, d_2 \quad n_2$ $\qquad$ $d_1 h(n) \lg n \leq f_2(n) \leq d_2 h(n) \lg n$ $\quad +$

chceme

$\exists a_1, a_2 \quad n_0$ $\qquad$ $a_1 h(n) \lg n \leq (f_1+f_2)(n) \leq a_2 h(n) \lg n$

$a_1 = d_1$ $\qquad$ $n_0 = \max\{n_1, n_2, 2\}$

$a_2 = c_2 + d_2$

$$\underbrace{c_1 \overset{>0}{h(n)} + d_1 h(n) \lg n}_{A} \le f_1(n) + f_2(n) \le \underbrace{c_2 h(n) + d_2 h(n) \lg n}_{B}$$

$$d_1 \underbrace{h(n) \lg n} \le A$$

$$B \le (c_2 + d_2)\, h(n) \lg n$$

$$\boxed{h(n) \le h(n) \lg n}$$

pro $n \geq 2$

$$\underline{c_2}\, h(n) + d_2 h(n) \lg n \le \underline{c_2 h(n) \lg(n)} + d_2 h(n) \lg n =$$

$$= (c_2 + d_2)\, h(n) \lg n$$

Z přednášky

- Pro každé $a > 1$ a $b > 1$ platí $\log_a(n) \in \Theta(\log_b(n))$.
- $\boxed{\lg n! \in \Theta(n \lg n).}$
- Věta (Gauss). Pro každé $n \geq 1$ platí $n^{n/2} \leq n! \leq \left(\frac{n+1}{2}\right)^n$.
- Věta Ať $f(n)$ je nezáporná neklesající funkce.

  Pokud $\underline{f(\frac{n}{2}) \in \Theta(f(n))}$, pak

  $\lg n$, n^2

  $$\sum_{k=1}^{n} f(k) \in \Theta(nf(n)).$$

  neplatí pro 2^n

$$\lim_{n\to\infty} \frac{\left(\frac{n}{2}\right)^2}{n^2} = \frac{1}{4}$$

$2^{\frac{n}{2}} \notin \Theta(2^n)$

$\lg(ab) = \lg a + \lg b$

$$\lg(n!) = \lg(1\cdot 2\cdot 3\cdot \ldots \cdot n) = \lg 1 + \lg 2 + \lg 3 + \ldots + \lg n$$

$$= \sum_{k=1}^{n} \lg k \in \Theta(n f(n))$$

$$\Theta(n \lg n)$$

$f(n) = \lg n$ je nezáporná, rostoucí

$f(\frac{n}{2}) = \lg \frac{n}{2} \in \Theta(\lg n)$ ~~?~~

$$\lg \frac{n}{2} = \lg n - \lg 2 = \lg n - 1$$

$$c_1 \lg n \leq \lg \frac{n}{2} \leq c_2 \lg n$$

$$\boxed{\begin{aligned} c_1 &= \tfrac{1}{2} \\ c_2 &= 1 \\ n_0 &= 4 \end{aligned}}$$

$\lg n - 1$

$c_2 = 1$

trn

$c_1 = \frac{1}{2}$

$\frac{1}{2}\lg n \leq \lg n - 1$

$1 \leq \frac{1}{2}\lg n$

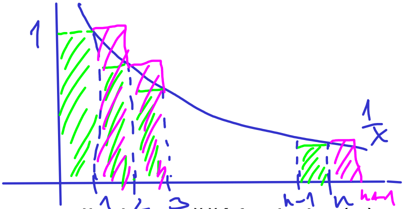
$2 \leq \lg n$, pro $n \geq 4$

$$c_1 \lg n + 1 \leq \lg n \leq c_2 \lg n + 1$$

$$\lim_{n\to\infty} \frac{\lg \frac{n}{2}}{\lg n} = 1$$

$$\int \frac{1}{x}\,dx = \ln x + c$$
$x > 0$

$$\sum_{k=1}^{n} h(k)$$



Je dána funkce $f(n) = \sum_{k=1}^{n} \frac{1}{k}$. Najděte co nejjednodušší funkci $g(n)$, pro kterou platí $f(n) \in \Theta(g(n))$.

$h(n) = \frac{1}{n} \geq 0$ je klesající

$h(x) = \frac{1}{x}$

$$[\ln x]_1^{n+1} = \int_1^{n+1} \frac{1}{x}\,dx \leq \sum_{k=1}^{n} \frac{1}{k} \leq \int_0^n \frac{1}{x}\,dx = [\ln x]_0^n = \infty$$

$$= \ln n - \ln 0^+$$
$$= \ln n - (-\infty)$$

$= \ln(n+1) - \ln 1$

$= \ln(n+1)$

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \sum_{k=2}^{n} \frac{1}{k} \leq 1 + \int_1^n \frac{1}{x}\,dx$$

$$= 1 + [\ln x]_1^n = 1 + \ln n - \underbrace{\ln 1}_{0}$$

$$\underbrace{c_1 \ln n \le \ln(n+1)}_{\ln n \text{ je rostoucí}} \le \sum_{k=1}^{n} \frac{1}{k} \le \underbrace{1 + \ln n \le c_2 \ln n}_{1 \le \ln n,\ n \ge 3}$$

$c_1 = 1$

$c_2 = 2$

$$g(n) = \ln n$$

$$\sum_{k=1}^{n} \frac{1}{k} \in \Theta(\ln n)$$

$$\sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1/3}{1-1/3} = \frac{1}{2}$$

$a_1 = \frac{1}{3}$ $q = \frac{1}{3}$

$$\frac{1}{3} \leq \sum_{k=1}^{n} \frac{1}{3^k} \leq \sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{2}$$

$$\sum_{k=1}^{n} \frac{1}{3^k} \in \Theta(1)$$

Nalezněte horní odhad

$$\sum_{k=1}^{n} \frac{k}{3^k} = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \dots + \frac{n}{3^n}$$

Hint: Pro horní odhad sumy použijte vhodnou geometrickou řadu.

$k=1$: $\frac{1}{3}$

$k=2$: $\frac{1}{3^2} + \frac{1}{3^2}$

$k=3$: $\frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3}$

$k=4$: $\frac{1}{3^4} + \frac{1}{3^4} + \frac{1}{3^4} + \frac{1}{3^4}$

$\vdots$ ...

$$\frac{k}{3^k} \leq \frac{k}{3}$$

$$\frac{k}{3^k} \leq \frac{n}{3^k}$$

$$\lim \frac{a_{n+1}}{a_n} < 1$$

$$\frac{1}{3}\cdot 1 \le \underbrace{\sum_{k=1}^{n} \frac{k}{3^k}}_{f(n)} \le \sum_{k=1}^{\infty} \frac{k}{3^k} = \sum_{k=1}^{\infty} \frac{1}{3^k} + \sum_{k=2}^{\infty} \frac{1}{3^k} + \sum_{k=3}^{\infty} \frac{1}{3^k} + \dots$$

$$= \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{1}{3^2}}{1-\frac{1}{3}} + \frac{\frac{1}{3^3}}{1-\frac{1}{3}} + \dots$$

$$= \frac{3}{2}\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots\right) = \frac{3}{4}\cdot 1$$

$$\sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{2}$$

$$\sum_{k=1}^{n} \frac{k}{3^k} \in \Theta(1)$$

new

Dokažte nebo vyvraťte: Jestliže pro neklesající nezáporné funkce $f(n)$ a $g(n)$ platí $f(n) \in \Theta(g(n))$ a $g(n) \in \Omega(1)$, pak pro každou konstantu $k > 0$ platí

$$f(n) + k \in \Theta(g(n)).$$

$\exists c_1, c_2 > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 \quad c_1 g(n) \overset{1}{\leq} f(n) \overset{2}{\leq} c_2 g(n)$

$\exists c_3 > 0 \ \exists n_1 \in \mathbb{N} \ \forall n \geq n_1 \quad g(n) \geq c_3 \cdot 1 \Rightarrow c_3 \leq g(n)$

$k = \frac{k}{c_3} \cdot c_3 \overset{3}{\leq} \frac{k}{c_3} g(n)$

chceme

$\exists d_1, d_2 > 0 \ \exists n_2 \in \mathbb{N} \ \forall n \geq n_2 \quad d_1 g(n) \leq f(n) + k \leq d_2 g(n)$

$c_1 g(n) \leq c_1 g(n) + \underbrace{k}_{>0} \overset{1}{\leq} f(n) + k \overset{2}{\leq} c_2 g(n) + k \overset{3}{\leq} c_2 g(n) + \frac{k}{c_3} g(n) =$

$= \left(c_2 + \frac{k}{c_3}\right) g(n)$

$d_1 = c_1$

$d_2 = c_2 + \frac{k}{c_3}$

$n_2 = \max\{n_0, n_1\}$

Tvrzení new neplatí pro klesající $f(n)$ a $g(n)$

pro $f(n) = \frac{1}{n}$, $g(n) = \frac{2}{n}$, $f(n) \in \Theta(g(n))$, ale
$f(n)+k \notin \Theta(g(n))$.

Opravdu, neex. $d > 0$ tak, že $f(n)+k \leq d\, g(n)$

$$\frac{1}{n} < \frac{1}{n} + k \leq d\frac{2}{n}$$

$$k \leq \frac{2d-1}{n}$$

$$n \leq \frac{2d-1}{k}$$ takové d neex.

Platí $f(n)+k \in \Theta(1)$:

$$k \leq \underbrace{f(n)}_{0 < \frac{1}{n} \leq 1} + k \leq 1+k$$

Dokažte nebo vyvraťte: Jestliže pro funkce $f(n)$ a $g(n)$ platí $f(n) \in \Theta(g(n))$ a $g(n) \in \Omega(1)$, pak

$$\sum_{i=1}^{n} f(i) \in \Theta(\sum_{i=1}^{n} g(i)).$$

$\exists c_3 > 0 \; \exists n_3 \in \mathbb{N} \; \forall n > n_3$
$g(n) \geq c_3$

Víme: $\exists c_1, c_2 > 0 \; \exists n_2 \in \mathbb{N} \; \forall n \geq n_2$ $\quad c_1 g(n) \leq f(n) \leq c_2 g(n)$

chceme $\exists d_1, d_2 > 0 \quad \exists n_1 \in \mathbb{N} \; \forall n \geq n_1$

$$d_1 \sum_{i=1}^{n} g(i) \leq \sum_{i=1}^{n} f(i) \leq d_2 \sum_{i=1}^{n} g(i)$$

$d_1 = \min\{c_1, k\}$
$d_2 = \max\{c_2, k\}$
$n_1 = n_0 = \max\{n_2, n_3\}$

Pro $i \geq n_0$ $\quad c_1 c_3 < c_1\, g(i) \leq f(i) \leq c_2\, g(i)$

Proto $c_1 c_3 (n-n_0) < c_1 \sum_{i=n_0}^{n} g(i) \leq \sum_{i=n_0}^{n} f(i) \leq c_2 \sum_{i=n_0}^{n} g(i)$ $\quad (*)$ $\quad +B$

$$\sum_{i=1}^{n} g(i) = \underbrace{\sum_{i=1}^{n_0-1} g(i)}_{A} + \sum_{i=n_0}^{n} g(i) \qquad A \neq 0$$

$$\sum_{i=1}^{n} f(i) = \underbrace{\sum_{i=1}^{n_0-1} f(i)}_{B} + \sum_{i=n_0}^{n} f(i) \qquad B \neq 0$$

$B = kA$

$k = \frac{B}{A}$

$$\leq c_1 \sum_{i=n_0}^{n} g(i) + k \sum_{i=1}^{n_0-1} g(i) \leq \sum_{i=1}^{n} f(i) \leq c_2 \sum_{i=n_0}^{n} g(i) + k \sum_{i=1}^{n_0-1} g(i) \leq$$

$c_2 \geq k \qquad c_2 \sum_{i=n_0}^{k} g(i) + k \sum_{i=1}^{n_0-1} g(i) \leq c_2 \sum_{i=1}^{n} g(i)$

$c_2 < k \qquad\qquad \leq k \sum_{i=1}^{n} g(i)$

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nebo | z ⊛ $\sum_{i=n_0}^{n} f(i) \in \Theta\left(\sum_{i=n_0}^{n} g(i)\right)$

$\sum_{i=1}^{n} f(i) = \underbrace{B}_{>0} + \sum_{i=n_0}^{n} f(i) \overset{\text{new}}{\in} \Theta\left(\sum_{i=n_0}^{n} g(i)\right)$

z 1. cvičení $\sum_{i=n_0}^{n} g(i) \in \Theta\left(\sum_{i=1}^{n} f(i)\right)$ a

$\sum_{i=1}^{n} g(i) = \underbrace{A}_{>0} + \sum_{i=n_0}^{n} g(i) \overset{\text{new}}{\in} \Theta\left(\sum_{i=1}^{n} f(i)\right)$

proto i $\sum_{i=1}^{n} f(i) \in \Theta\left(\sum_{i=1}^{n} g(i)\right)$

$$\sum_{k=1}^{n} \frac{1}{k}$$

Odhadněte asymptotický růst funkcí:

1. $\sum_{i=1}^{n} \frac{1}{i^{1,1}}$. integrál.
2. $\sum_{i=1}^{n} (8\frac{i}{2^i})$. $\sum \frac{k}{3^k}$
3. $\sum_{i=1}^{n} (i^4 \lg^3 i + i^3 \lg^9 i)$.

$\Theta(n^5 \lg^3 n)$

$\sum_{i=1}^{n} g(i) \in \Theta(n\, g(n))$

$f(n) = n^4 \lg^3 n + n^3 \lg^9 n \in \Theta(g(n))$

$\boxed{g(n) = n^4 \lg^3 n}$

$g(n) \geq 0$ rostoucí

$g(\frac{n}{2}) \in \Theta(g(n))$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty}\left(1 + \frac{n^3 \lg^9 n}{n^4 \lg^3 n}\right) = \lim_{n\to\infty}\left(1 + \frac{\lg^6 n}{n}\right) = 1$$

$$\lim_{n\to\infty} \frac{(\frac{n}{2})^4 \lg^3 \frac{n}{2}}{n^4 \lg^3 n} = \left(\frac{1}{2}\right)^4$$

Odhadněte asymptotický růst $\sum_{i=1}^{n} \frac{1}{i^{1,1}} \in \Theta(1)$ $f(n) = \frac{1}{n^{1,1}}$ klesající

$$\int_1^{n+1} x^{-1,1}\,dx \le \sum_{i=1}^{n} \frac{1}{i^{1,1}} \le \frac{1}{1^{1,1}} + \int_1^{n} x^{-1,1}\,dx$$

$$\int x^{-1,1}\,dx = \frac{x^{-0,1}}{-0,1} + c = -10\,\frac{1}{x^{0,1}} + C$$

$$\int_1^{n+1} x^{-1,1}\,dx = \left[-10\,\frac{1}{x^{0,1}}\right]_1^{n+1} = 10 - \frac{10}{(n+1)^{0,1}} \ge 5 \text{ pro dost velká } n$$

$\frac{10}{(n+1)^{0,1}} \xrightarrow{n\to\infty} 0$

$$1 + \int_1^{n} x^{-1,1}\,dx = 1 + \left[-10\,\frac{1}{x^{0,1}}\right]_1^{n} = 1 + 10 - \underbrace{\frac{10}{n^{0,1}}}_{>0} \le 11$$

Odhadněte asymptotický růst $\sum_{i=1}^{n} (8\frac{i}{2^i})$. $= 8 \sum_{i=1}^{n} \frac{i}{2^i} \in \Theta(1)$

$$\frac{1}{2} \leq \sum_{i=1}^{n} \frac{i}{2^i} \leq \sum_{i=1}^{\infty} \frac{i}{2^i} = \sum_{i=1}^{\infty} \frac{1}{2^i} + \sum_{i=2}^{\infty} \frac{1}{2^i} + \sum_{i=3}^{\infty} \frac{1}{2^i} + \dots$$

$i=1$: $\frac{1}{2}$

$i=2$: $\frac{1}{2^2} + \frac{1}{2^2}$

$i=3$: $\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3}$

$i=4$: $\frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4}$

$$= \frac{1/2}{1-1/2} + \frac{1/2^2}{1-1/2} + \frac{1/2^3}{1-1/2} + \dots$$

$$= 2 \cdot \sum_{i=1}^{\infty} \frac{1}{2^i} = 2$$

$$4 \cdot 1 \leq \sum_{i=1}^{n} \left(8\frac{i}{2^i}\right) \leq 16 \cdot 1$$