

# Teorie algoritmů

Natalie Žukovec

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$$\lim_{n \rightarrow \infty} \frac{5^n + n^4}{5^n} = \lim_{n \rightarrow \infty} \frac{5^n}{5^n} + \lim_{n \rightarrow \infty} \frac{n^4}{5^n} = 1 + 0 = 1$$

Z definice ukaŹte, Źe platí: Pro  $f(n) = 5^n + n^4$  máme  $f(n) \in \Theta(5^n)$ .

$$\exists c_1, c_2 > 0 \exists n_1 \in \mathbb{N} \forall n \geq n_1 \quad c_1 5^n \leq 5^n + n^4 \leq c_2 5^n$$

$$c_1 = 1 \in \mathbb{R}$$

$$c_2 = 2$$

$$n_1 = 1$$

$$c_1 = 1, \text{ protože } n^4 \geq 0$$

$$c_2 = 2$$

$$5^n + n^4 \leq 2 \cdot 5^n$$

$$n^4 \leq 5^n$$

$$\text{pro } n \geq 1$$

Jsou dány tři nezáporné funkce  $f(n)$ ,  $g(n)$  a  $h(n)$ . Dokažte:

Jestliže  $f(n)$  je  $\mathcal{O}(g(n))$  a  $g(n)$  je  $\mathcal{O}(h(n))$ , tak  $f(n)$  je  $\mathcal{O}(h(n))$ .

$$\left[ \begin{array}{l} \exists c_1 > 0 \exists n_1 \in \mathbb{N} \forall n \geq n_1 \quad f(n) \leq c_1 g(n) \\ \exists c_2 > 0 \exists n_2 \in \mathbb{N} \forall n \geq n_2 \quad c_1 g(n) \leq c_2 h(n) \end{array} \right. \left. \begin{array}{l} \\ \\ \end{array} \right]$$

chceme

$$\exists c_3 > 0 \exists n_3 \in \mathbb{N} \forall n \geq n_3 \quad f(n) \leq c_3 h(n)$$

$$c_3 = c_1 c_2$$

$$n_3 = \max\{n_1, n_2\}$$

$$f(n) \leq c_1 g(n) \leq c_1 c_2 h(n)$$

$$(f_1 + f_2)(n) = f_1(n) + f_2(n)$$

Dokažte následující tvrzení:

Jsou dány nezáporné funkce  $f_1(n)$ ,  $f_2(n)$  a  $h(n)$  takové, že

$f_1(n) \in \Theta(h(n))$ ,  $f_2(n) \in \Theta(h(n) \lg n)$ , pak  $(f_1 + f_2)(n) \in \Theta(h(n) \lg n)$ .

$$\exists c_1 c_2 n_1$$
$$\exists d_1 d_2 n_2$$

$$c_1 h(n) \leq f_1(n) \leq c_2 h(n)$$
$$d_1 h(n) \lg n \leq f_2(n) \leq d_2 h(n) \lg n \quad \oplus$$

chceme

$$\exists a_1 a_2 n_0$$

$$a_1 h(n) \lg n \leq (f_1 + f_2)(n) \leq a_2 h(n) \lg n$$

$$a_1 = d_1$$

$$a_2 = c_2 + d_2$$

$$n_0 = \max \{ n_1, n_2, 2 \}$$

$$\underbrace{c_1 h(n) + d_1 h(n) \lg n}_{A} = f_1(n) + f_2(n) \leq \underbrace{c_2 h(n) + d_2 h(n) \lg n}_B$$

$$d_1 h(n) \lg n \leq A$$

$$B \leq (c_2 + d_2) h(n) \lg n$$
$$\underbrace{h(n) \leq h(n) \lg n}_{\text{pro } n \geq 2}$$

$$\underbrace{c_2 h(n) + d_2 h(n) \lg n} \leq \underbrace{c_2 h(n) \lg(n) + d_2 h(n) \lg n}_{(c_2 + d_2) h(n) \lg n}$$

## Z přednášky

- ▶ Pro každé  $a > 1$  a  $b > 1$  platí  $\log_a(n) \in \Theta(\log_b(n))$ .
- ▶  $\lg n! \in \Theta(n \lg n)$ .
- ▶ Věta (Gauss). Pro každé  $n \geq 1$  platí  $n^{n/2} \leq n! \leq \left(\frac{n+1}{2}\right)^n$ .
- ▶ Věta Ať  $f(n)$  je nezáporná neklesající funkce.

Pokud  $f\left(\frac{n}{2}\right) \in \Theta(f(n))$ , pak

$\lg n, n^2$   
neplatí pro  $2^n$

$$2^{\frac{n}{2}} \notin \Theta(2^n)$$

$$\sum_{k=1}^n f(k) \in \Theta(nf(n)).$$

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{n}{2}\right)^2}{n^2} = \frac{1}{4}$$

$$\lg(ab) = \lg a + \lg b$$

$$\lg(n!) = \lg(1 \cdot 2 \cdot 3 \cdot \dots \cdot n) = \lg 1 + \lg 2 + \lg 3 + \dots + \lg n$$

$$= \sum_{k=1}^n \lg k \in \Theta(n f(n))$$

$$\Theta(n \lg n)$$

$f(n) = \lg n$  je nerávná, roztačí

$$f\left(\frac{n}{2}\right) = \lg \frac{n}{2} \in \Theta(\lg n) \quad \times$$

$$\lg \frac{n}{2} = \lg n - \lg 2 = \lg n - 1$$

$$c_1 = \frac{1}{2}$$

$$c_2 = 1$$

$$n_0 = 4$$

$$c_1 = \frac{1}{2}$$

$$c_1 \lg n \leq \lg \frac{n}{2} \leq c_2 \lg n$$

$$c_2 = 1$$

$$n$$

$$\frac{1}{2} \lg n \leq \lg n - 1$$

$$1 \leq \frac{1}{2} \lg n$$

$$2 \leq \lg n, \text{ pro } n \geq 4$$

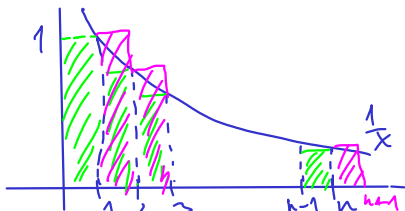
$$c_1 \lg n + 1 \leq \lg n \leq c_2 \lg n + 1$$

$$\lim_{n \rightarrow \infty} \frac{\lg \frac{n}{2}}{\lg n} = 1$$

$$\int \frac{1}{x} dx = \ln x + c$$

$x > 0$

$$\sum_{k=1}^n h(k)$$



Je dána funkce  $f(n) = \sum_{k=1}^n \frac{1}{k}$ . Najděte co nejjednodušší funkci  $g(n)$ , pro kterou platí  $f(n) \in \Theta(g(n))$ .

$h(k) = \frac{1}{k} \geq 0$  je klesající

$$h(x) = \frac{1}{x}$$

$$\ln(n+1) - \ln 1 = \int_1^{n+1} \frac{1}{x} dx \leq \sum_{k=1}^n \frac{1}{k} \leq \int_0^n \frac{1}{x} dx = [\ln x]_0^n = \ln n - \ln 0^+ = \ln n - (-\infty)$$

$$\ln(n+1) - \ln 1$$

$$\ln(n+1)$$

$$\sum_{k=1}^n \frac{1}{k} = 1 + \sum_{k=2}^n \frac{1}{k} \leq 1 + \int_1^n \frac{1}{x} dx = 1 + [\ln x]_1^n = 1 + \ln n - \ln 1 = 1 + \ln n$$



$$c_1 \ln n \leq \ln(n+1) \leq \sum_{k=1}^n \frac{1}{k} \leq 1 + \ln n \leq c_2 \ln n$$

$$c_1 = 1$$

$$c_2 = 2$$

$\ln n$  je zostava

$$g(n) = \ln n$$

$$1 \leq \ln n$$
$$n \geq 3$$

$$\sum_{k=1}^n \frac{1}{k} \in \Theta(\ln n)$$

$$\sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1/3}{1-1/3} = \frac{1}{2}$$

$a_1 = \frac{1}{3} \quad r = \frac{1}{3}$

Nalezněte horní odhad

$$\frac{1}{3} \leq \sum_{k=1}^n \frac{1}{3^k} \leq \sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{2}$$

$$\sum_{k=1}^n \frac{1}{3^k} \in \mathcal{O}(1)$$

$$\sum_{k=1}^n \frac{k}{3^k} = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \dots + \frac{n}{3^n}$$

Hint: Pro horní odhad sumy použijte vhodnou geometrickou řadu.

$k=1$   $\frac{1}{3}$   
 $k=2$   $\frac{1}{3^2} + \frac{1}{3^2}$   
 $k=3$   $\frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3}$   
 $k=4$   $\frac{1}{3^4} + \frac{1}{3^4} + \frac{1}{3^4} + \frac{1}{3^4}$   
 ...

$$\frac{k}{3^k} \leq \frac{k}{3}$$

$$\frac{k}{3^k} \leq \frac{n}{3^k}$$

$$\lim \frac{a_{n+1}}{a_n} < 1$$

$$\left( \frac{1}{3} \cdot 1 \right) \leq \sum_{k=1}^n \frac{k}{3^k} \leq \sum_{k=1}^{\infty} \frac{k}{3^k} = \sum_{k=1}^{\infty} \frac{1}{3^k} + \sum_{k=2}^{\infty} \frac{1}{3^k} + \sum_{k=3}^{\infty} \frac{1}{3^k} + \dots$$

$$= \frac{\frac{1}{3}}{1 - \frac{1}{3}} + \frac{\frac{1}{3^2}}{1 - \frac{1}{3}} + \frac{\frac{1}{3^3}}{1 - \frac{1}{3}} + \dots$$

$$= \frac{3}{2} \left( \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots \right) = \frac{3}{4} \cdot 1$$

$$\sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{2}$$

$$\sum_{k=1}^n \frac{k}{3^k} \in \Theta(1)$$

Dokažte nebo vyvráťte: Jestliže pro funkce  $f(n)$  a  $g(n)$  platí  $f(n) \in \Theta(g(n))$  a  $g(n) \in \Omega(1)$ , pak

$$\sum_{i=1}^n f(i) \in \Theta\left(\sum_{i=1}^n g(i)\right).$$

$$\exists c_3 > 0 \exists n_3 \in \mathbb{N} \forall n > n_3 \\ g(n) \geq c_3$$

Víme:  $\exists c_1, c_2 > 0 \exists n_2 \in \mathbb{N} \forall n \geq n_2$   $c_1 g(n) \leq f(n) \leq c_2 g(n)$

chtíme  $\exists d_1, d_2 > 0 \exists n_1 \in \mathbb{N} \forall n \geq n_1$

$$d_1 = \min\{c_1, k\}$$

$$d_2 = \max\{c_2, k\}$$

$$n_1 = n_0 = \max\{n_2, n_3\}$$

$$d_1 \sum_{i=1}^n g(i) \leq \sum_{i=1}^n f(i) \leq d_2 \sum_{i=1}^n g(i)$$

$$\text{Pro } i \geq n_0 \quad c_1 c_3 < c_1 \quad g(i) \leq f(i) \leq c_2 g(i)$$

$$\text{Proto } c_1 c_3^{(n-n_0)} < c_1 \sum_{i=n_0}^n g(i) \leq \sum_{i=n_0}^n f(i) \leq c_2 \sum_{i=n_0}^n g(i) \quad | \quad +B$$

$$\sum_{i=1}^n g(i) = \underbrace{\sum_{i=1}^{n_0-1} g(i)}_A + \underbrace{\sum_{i=n_0}^n g(i)}_{A \neq 0}$$

$$\sum_{i=1}^n f(i) = \underbrace{\sum_{i=1}^{n_0-1} f(i)}_B + \underbrace{\sum_{i=n_0}^n f(i)}_{B \neq 0}$$

$$B = kA$$

$$k = \frac{B}{A}$$

$$\leq c_1 \sum_{i=n_0}^n g(i) + k \sum_{i=1}^{n_0-1} g(i) \leq \sum_{i=1}^n f(i) \leq c_2 \sum_{i=n_0}^n g(i) + k \sum_{i=1}^{n_0-1} g(i) \leq$$

$$c_2 \geq k$$

$$c_2 \sum_{i=0}^k g(i) + k \sum_{i=1}^{n_0-1} g(i) \leq c_2 \sum_{i=1}^n g(i)$$

$$c_2 < k$$

$$\leq k \sum_{i=1}^n g(i)$$

$$\sum_{k=1}^n \frac{1}{k}$$

Odhadněte asymptotický růst funkcí:

1.  $\sum_{i=1}^n \frac{1}{i^{1.1}}$ . integrál.

2.  $\sum_{i=1}^n (8 \frac{i}{2^i})$ .  $\sum \frac{k}{3^k}$

3.  $\sum_{i=1}^n (i^4 \lg^3 i + i^3 \lg^9 i)$ .

$$\Theta(n^5 \lg^3 n)$$

$$\sum_{i=1}^n g(i) \in \Theta(n g(n))$$

$$f(n) = n^4 \lg^3 n + n^3 \lg^9 n \in \Theta(n^4 \lg^3 n)$$

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \left( 1 + \frac{n^3 \lg^9 n}{n^4 \lg^3 n} \right) = \lim_{n \rightarrow \infty} \left( 1 + \frac{\lg^6 n}{n} \right) = 1$$

$$\lim_{n \rightarrow \infty} \frac{(\frac{n}{2})^4 \lg^3 \frac{n}{2}}{n^4 \lg^3 n} = \left(\frac{1}{2}\right)^4$$

$$g(n) = n^4 \lg^3 n$$

$g(n) \geq 0$   
rostoucí  
 $g(\frac{n}{2}) \in \Theta(g(n))$

Odhadněte asymptotický růst  $\sum_{i=1}^n \frac{1}{i^{1,1}} \in \Theta(1)$   $f(n) = \frac{1}{n^{1,1}}$  klesající

$$\int_1^{n+1} x^{-1,1} dx \leq \sum_{i=1}^n \frac{1}{i^{1,1}} \leq \frac{1}{1^{1,1}} + \int_1^n x^{-1,1} dx$$

$$\int x^{-1,1} dx = \frac{x^{-0,1}}{-0,1} + C = -10 \frac{1}{x^{0,1}} + C$$

$$\int_1^{n+1} x^{-1,1} dx = \left[ -10 \frac{1}{x^{0,1}} \right]_1^{n+1} = 10 - \frac{10}{(n+1)^{0,1}} \geq 5 \text{ pro dost velká } n$$

$$1 + \int_1^n x^{-1,1} dx = 1 + \left[ -10 \frac{1}{x^{0,1}} \right]_1^n = 1 + 10 - \frac{10}{n^{0,1}} \leq 11$$



Odhadněte asymptotický růst  $\sum_{i=1}^n (8 \frac{i}{2^i}) = 8 \sum_{i=1}^n \frac{i}{2^i} \in \mathcal{O}(1)$

$$\frac{1}{2} \leq \sum_{i=1}^n \frac{i}{2^i} \leq \sum_{i=1}^{\infty} \frac{i}{2^i} = \sum_{i=1}^{\infty} \frac{1}{2^i} + \sum_{i=2}^{\infty} \frac{1}{2^i} + \sum_{i=3}^{\infty} \frac{1}{2^i} + \dots$$

$$\begin{array}{l} i=1 \\ i=2 \\ i=3 \\ i=4 \end{array} \quad \begin{array}{l} \frac{1}{2} \\ \frac{1}{2^2} + \frac{1}{2^2} \\ \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3} \\ \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} \end{array}$$

$$= \frac{1/2}{1-1/2} + \frac{1/2^2}{1-1/2} + \frac{1/2^3}{1-1/2} + \dots$$

$$= 2 \cdot \sum_{i=1}^{\infty} \frac{1}{2^i} = 2$$

$$4 \cdot 1 \leq \sum_{i=1}^n (8 \frac{i}{2^i}) \leq 16 \cdot 1$$